SOME RESULTS IN CHIRAL ANOMALY, TRACE ANOMALY AND ENERGY MOMENTUM TENSOR

by

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DEPARTMENT OF PHYSICS

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SOME RESULTS IN CHIRAL ANOMALY, TRACE ANOMALY AND ENERGY MOMENTUM TENSOR

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ANURADHA MISRA

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Certified that the work contained in this thesis entitled "SOME RESULTS IN CHIRAL ANOMALY, TRACE ANOMALY AND ENERGY MOMENTUM TENSOR" has been carried out by Ms.Anuradha Misra under my supervision and has not been submitted elsewhere for a degree.

Kanpur

January 25,1989.

S D Joglekar

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—— Anuradha Misra

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SYNOPSIS

SOME RESULTS IN CHIRAL ANOMALY, TRACE ANOMALY AND ENERGY MOMENTUM TENSOR

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The path integral formulation of field theory has led to many of the most important developments in theoretical physics over the past several decades. However, the anomalies appearing in the Ward identities were not explained in this formalism till 1979 when Fujikawa, in a remarkable series of papers, derived almost all known anomalies as the change in functional integral measure under the corresponding symmetry transformation. However, Fujikawa's derivations which have occasionally been referred to as non-perturbative are, in fact, valid only to one loop order.

A part of this work is concerned with extending Fujikawa's derivation of the chiral anomaly to all orders in perturbation theory. Fujikawa's formalism consists of regularizing the ill-defined anomaly term by using a cutoff M on large eigenvalues of the operator $\not{D}$ and taking the limit $M^2 \longrightarrow \infty$ at the end of the calculation. However, as we have argued, the

operation of interchanging the limit $M^2 \longrightarrow \infty$ and performing the functional integral (which is implicit in Fujikawa's derivation) is not valid beyond one loop order. The expression for the regularized anomaly can be expanded as a power series in $1/M^2$ (the first term of the series yielding the chiral anomaly) and beyond one loop order all the terms in this series may contribute to the anomaly. To prove the Adler-Bardeen theorem in this context, one must take into account the contributions coming from the Green's functions of these higher order terms also.

We have obtained the general form of these higher order terms and then, by considerations of their divergence structure, we have obtained a form for the chiral anomaly equation which is valid to all orders in perturbation theory. Finally, we have presented an argument, not entirely within path integral framework, for the Adler-Bardeen theorem.

The second part of this work deals with the problem of obtaining a finite energy momentum tensor in theories involving scalars. One can always introduce new infinite counterterms to make matrix elements of the energy momentum tensor finite, but this amounts to introducing additional parameters to the theory in the presence of gravity apart from the usual parameters of the flat space theory. However, if one improves the energy momentum tensor in accordance with a finite improvement program, then no independent renormalization is needed to renormalize the energy momentum tensor and hence, no extra parameter from the experiment is required. In $\lambda\phi^4$-theory, a

finite improvement program was given by Collins in which the coefficient of the improvement term (which is added to make the energy momentum tensor finite) is a finite function of renormalized parameters and $\varepsilon(=4-n)$. This improved energy momentum tensor was found to be finite to all orders and the improvement coefficient was actually a function of ε only.

In this work, we have given a new interpretation of finite improvement program in which the improvement coefficient is a finite function of bare quantities. This kind of improved energy momentum tensor is more desirable than the one given by Collins (i.e. when the improvement coefficient is a finite function of renormalized quantities) because it is derivable from an action which is a finite function of bare quantities (as an action should be) and no extra renormalization condition (independent or not) is needed at all. Collins' improvement coefficient, being a function of ε only, is a special case of this kind of improvement coefficient also. We have proved a uniqueness theorem stating that Collins' improved energy momentum tensor is the unique energy momentum tensor of this special kind also.

We have also considered the possibility of having a finite improvement program in theories involving scalars and having two coupling constants. We have considered four such theories :

(i) Scalar Quantum Electrodynamics

(ii) Non abelian gauge theories with scalars

(iii) Yukawa theory

(iv) A theory of two interacting scalar fields.

We have proved in the context of each of these theories that it is impossible to make the matrix elements of the energy momentum tensor finite by adding to it an improvement term of either of the following two kinds:

(1) One, in which the improvement coefficient is a finite function of bare quantities

(2) One, in which the improvement coefficient is a finite function of renormalized quantities.

The implication of this negative result is that in these theories an extra piece of experimental information is necessarily needed to describe the theory in the presence of gravity.

CHAPTER 1

INTRODUCTION

Anomalies were first discovered by applying perturbation theory methods to the calculation of various physical amplitudes[1]. The famous triangle diagram contributes to the two photon matrix elements of the axial vector current when computed in one loop approximation. Adler[2] was the first to show that the anomalous breakdown of the axial vector Ward identity could be described by replacing the naive expression $2i m_o \bar{\psi}\gamma_5\psi$ of the divergence of the axial vector current by

$$\partial_\mu J^\mu_5 = 2i m_o \bar{\psi}\gamma_5\psi + \frac{e_o^2}{8\pi^2} F\tilde{F}$$

It was shown later by Adler and Bardeen[3] that the anomalous term in this equation receives contributions only from the one loop matrix elements, i.e. higher loop diagrams do not contribute to the chiral anomaly. The original statement given in the context of spinor electrodynamics for the Adler Bardeen non renormalization theorem makes use of a cutoff. These non renormalization theorems have been proved in various regularization schemes and also by the use of renormalization group methods[4]. However, in the path integral framework, where the anomalies have been obtained by Fujikawa[5] as a change in the functional integral measure, this aspect of higher order corrections to the anomaly equation has,

unfortunately, been ignored. Non-renormalization of chiral anomaly has sometimes given rise to the misconception that the results obtained by Fujikawa's procedure are exact. However, these results hold only to one loop order[6,7].

In the first part of this thesis, we shall be concerned with the non-renormalization of chiral anomaly within Fujikawa's path integral framework. In this formalism, as stated earlier, the anomaly arises as the Jacobian factor for chiral transformation. To obtain this change in measure, Fujikawa expands the fermion fields in terms of a complete set of basis functions $\phi_n(x)$ which are actually eigenfunctions of the gauge invariant, hermitian operator $\not{D}$ in Euclidean space. In this basis, the change in the functional integral measure involves $\sum_n \phi^\dagger(x)\gamma_5\phi_n(x)$ which, being an ill-defined object, is regularized by using a cutoff such as $\exp(-\lambda_n^2/M^2)$, where λ_n's are the eigenvalues of $\not{D}$. Fujikawa expands the expression for regularized anomaly in powers of M^{-2}, evaluates the M-independent part as the chiral anomaly and then takes the limit $M^2 \longrightarrow \infty$, thus neglecting the higher order terms in the series in M^{-2}. However, this limit is to be taken after performing a functional integral and it is quite possible that the Green's functions of these higher order terms have divergences of sufficiently high order in M^2, so that the contribution coming from them may be non vanishing (as elaborated in Chapter 2).Thus, to prove the Adler Bardeen theorem, one must take into account all the terms in this

series.

We have had a partial success in this direction by obtaining a general form of these terms and in obtaining from that a form for the anomaly equation which is valid to <u>all</u> orders in perturbation theory. We have also given an argument for the Adler Bardeen theorem by using a result from Pauli Villars regularization.

In the remaining part of the thesis, we have dealt with a finite improvement program, which is used in scalar theories to obtain finite matrix elements for energy momentum tensor without introducing any independent renormalizations.

Renormalization of energy momentum tensor has been a subject of keen interest due to its physical significance[8-16]. Apart from the usual canonical methods, the energy momentum tensor can also be obtained from the curved space action $S[\phi,g]$ using

$$\theta_{\mu\nu} = \frac{2}{\sqrt{-g}} \left. \frac{\delta S[\phi,g]}{\delta g^{\mu\nu}} \right|_{g^{\mu\nu}=\eta^{\mu\nu}}$$

This energy momentum tensor, which is the source of gravitation, is related to physical processes — its matrix elements represent scattering in the presence of a weak external gravitational field— and hence finiteness is an important issue[10]. In theories involving scalars, the energy momentum tensor derived from the minimal Einstein action does not have finite matrix elements even to one loop order. However, one may always add to this action non-minimal terms which are consistent with the principle of general covariance.

There are such non-minimal terms which lead to energy momentum tensors containing improvement terms not present in the energy momentum tensor obtained via minimal Einstein action. But such terms are allowed so long they do not alter the conserved quantities. This idea of improving the energy momentum tensor so as to make its matrix elements finite has been pursued by various authors[8-15].

The advantage of obtaining finite matrix elements by using a finite improvement program over the usual method of renormalizing $\theta_{\mu\nu}$ by adding infinite counterterms is that in the former case there is no need for any additional experimental data in the presence of gravity, whereas in the latter case, the parameters of the flat space theory are no longer sufficient and some extra experimental information is necessarily required to define matrix elements for processes in presence of weak gravitational field[16-20].

It has been shown by Collins[13] that a finite improvement program exists in $\lambda\phi^4$-theory, which gives finite matrix elements to all orders in perturbation theory. However, the general form of the energy momentum tensor suggested by Collins is not derivable from an action which is a finite function of bare quantities. Thus, the action contains additional infinite counterterms apart from the usual renormalization counterterms of the flat space theory(although these counterterms are determined by the theory and no extra experimental input is needed).

In the present thesis, we emphasize on the desirability of a different kind of improvement program in which the improved energy momentum tensor is derivable from an action which is a finite function of bare quantities. Such an energy momentum tensor does not introduce any infinite counterterms in the action and no extra renormalization condition (independent or not) is needed[19].

As will be explained later in detail, there can be two different interpretations of the finite improvement program:

TYPE I   Improvement program, in which the improvement coefficient is a finite function of bare quantities.

TYPE II Improvement program, in which the improvement coefficient is a finite function of renormalized quantities.

In $\lambda\phi^4$-theory, a finite energy momentum tensor of type II has already been shown to exist by Collins[13]. This energy momentum tensor is, in fact, a function of ε only and is the unique energy momentum tensor of type II. However, this energy momentum tensor is a special case of type I improvement also and hence the existence of a successful type I finite improvement program is already established. However, there may be other energy momentum tensors of type I which may also have finite matrix elements. We have proved a uniqueness theorem regarding this, which states that an energy momentum tensor of type I, if finite to all orders, is the unique energy momentum tensor of this kind[16].

In the later part of the thesis, we have dealt with the possibility of having a finite improvement program in scalar theories having two coupling constants and additional fields. We shall prove, in the context of four such theories, that it is, in fact, impossible to have a finite improvement program of either type I or type II[17-20].

The plan of the thesis is as follows:

In chapter 2, we shall deal with the problem of obtaining higher order contributions to the chiral anomaly within Fujikawa's framework. In chapter 3, we shall discuss various interpretations of the finite improvement programs and stress on the significance of type I improvement program. In chapter 4, we shall prove the uniqueness theorem stated above for the $\lambda\phi^4$-theory. We will also show that the improved energy momentum tensor of type I can be derived using canonical methods from a bare action containing second derivatives of fields. In chapter 5, we shall deal with the energy momentum tensor of scalar Quantum Electrodynamics and will establish the impossibility of having a finite improvement program of either type I or type II. In chapter 6, we shall establish the same negative result for three other theories involving scalars and having two coupling constants. These theories are — non-abelian gauge theories with scalars, Yukawa theory and a theory of two interacting scalar fields.

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CHAPTER 2

NON RENORMALZATION OF CHIRAL ANOMALY

[2.1]INTRODUCTION

In any classical field theory, corresponding to every continuous symmetry of the Lagrangian there exists a conservation law given by Noether's theorem. These local conservation laws give rise to certain relations among Green's functions of the theory. Such relations among Green's functions following from the symmetry properties of the Lagrangian are called Ward identities. These identities play a crucial role in the derivation of current algebra low energy theorems and are also essential in the renormalization programme of any theory with non trivial symmetries. Therefore, it is important to check that these identities are not spoiled by higher order correction terms in perturbation theory. Indeed, there are situations in field theory, where a (classical) local conservation law derived from gauge invariance with the aid of Noether's theorem holds at tree level but is not respected by loop diagrams. Terms that violate a classical conservation law are known as anomalies. As an example, consider the Lagrangian of Quantum Electrodynamics(Q.E.D):

$$\mathcal{L} = \bar{\psi}(x)(\iota \not{D}-m)\psi(x) - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} \tag{2.1}$$

which is invariant upto mass terms under the local chiral

transformation

$$\psi(x) \longrightarrow \exp(\iota\alpha\,\gamma_5)\psi(x)$$
$$\bar{\psi}(x) \longrightarrow \bar{\psi}(x)\exp(\iota\alpha\,\gamma_5) \tag{2.2}$$

Noether's theorem leads to the following divergence equation

$$\partial_\mu J_5^\mu = 2m_o \iota\bar{\psi}(x)\gamma_5\psi(x) \tag{2.3}$$

where $J_5^\mu = \bar{\psi}\ \gamma^\mu\gamma_5\psi$ is the axial vector current.This conservation equation leads to the following axial vector Ward identity relating 3-point Green's functions

$$q^\lambda T_{\mu\nu\lambda} = 2mT_{\mu\nu} \tag{2.4}$$

where

$$T_{\mu\nu\lambda}(k_1,k_2,q) = \iota\int d^n x_1 d^n x_2 \langle 0|T(J_\mu(x_1)J_\nu(x_2)J_\lambda{}^5(0)|0\rangle e^{(\iota k_1 x_1+\iota k_2 x_2)}$$
$$T_{\mu\nu}(k_1,k_2,q) = \iota\int d^n x_1 d^n x_2 \langle 0|T(J_\mu(x_1)J_\nu(x_2)J\ ^5(0)|0\rangle e^{(\iota k_1 x_1+\iota k_2 x_2)} \tag{2.5}$$

$J_\mu(x)$ and $J^5(x)$ are respectively the vector and pseudoscalar currents given by

$$J_\mu(x) = \bar{\psi}(x)\gamma_\mu\psi(x) \quad ; \quad J_5(x) = \bar{\psi}(x)\gamma_5\psi(x) \tag{2.6}$$

and $q = k_1+k_2$.

However, when one calculates the lowest order contribution to $T_{\mu\nu\lambda}$ and $T_{\mu\nu}$ coming from the triangle diagrams,it is found that the naive Ward identity of Eq.(2.4) is not satisfied. These diagrams are superficially linearly divergent and hence are ill-defined.Adler[1] calculated these diagrams using a cutoff regularization and obtained the following anomalous Ward identity,

$$q^{\lambda} T_{\mu\nu\lambda} = 2mT_{\mu\nu} - \frac{1}{2\pi^2} \varepsilon_{\mu\nu\rho\sigma} k_1^{\rho} k_2^{\sigma} \qquad (2.7)$$

This leads to a modification of axial vector current divergence equation as

$$\partial_{\mu} J^{\mu}_5 = 2m_o i \bar{\psi}(x)\gamma_5 \psi(x) - \frac{1}{16\pi^2} \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu}(x) F_{\rho\sigma}(x) \qquad (2.8)$$

The second term on the right hand side discovered by Adler[1] and also by Bell and Jackiw[2] in their current algebra studies is the anomalous term known as ABJ or chiral anomaly.

ABJ anomaly arises due to higher order corrections. However, the anomalous term in Eq.(2.8) has been derived only in lowest order. One may expect a further modification of anomaly when corrections to Eq.(2.8) in orders higher than the lowest non-trivial order are taken into account. This is true in certain other cases e.g.the trace anomaly, but for the chiral anomalies the coefficient of anomaly is not affected by higher order radiative corrections.Adler and Bardeen[3] showed that the triangle diagrams with more than one loop do not contribute to the anomaly term. This statement of non renormalization of chiral anomaly is in particular the content of Adler Bardeen theorem.

ABJ anomalies may occur whenever there are fermions in the theory. Typically, this happens when a proper vertex involving an odd number of axial vector currents cannot be regularized in a way that preserves all the Ward Takahashi (WT) identities on such a vertex, and as a result some of the WT identities have to be broken. The reason for occurence of these anomalies is not the inability to devise a proper regularization scheme.In

certain theories such a scheme is impossible to devise.

Originally, Adler had derived the chiral anomaly in Eq.(2.8) using conventional cutoff regularization[1]. One may question how the anomalous term appears in other regularization schemes. In dimensioinal regularization, the problem shows up as the difficulty in defining γ_5 in space time dimension other than four[4].Derivations of anomalies using Pauli-Villars regularization [5] and point splitting regularization[6] are also well known. We shall be concerned here only with the path integral derivation of chiral anomaly. In path integral formulation, as shown by Fujikawa[7,8] the anomaly arises as the change in the functional integral measure. In Fujikawa formalism, Jacobian is an ill-defined quantity. Fujikawa regularizes this quantity in terms of eigenvalues of the operator $\not{D}$ and evaluates the anomaly by taking the limit $M^2 \longrightarrow \infty$ in the end, where M is the regulator mass. Fujikawa derived all known anomalies by this procedure[7-13]. Since then, various authors have offered modified or alternative versions of his results[14-17].

Fujikawa's derivations of anomalies have occasionally been labelled as non-perturbative[14,18]. This label has generally been applied to chiral anomaly in various theories. Non renormalization of chiral anomaly allows one to maintain this misconception that these results are exact. However the, results obtained in this approach are true only in one loop approximation. There are a number of reasons for this:

(i)One does not know how to calculate any quantity in Q.C.D. exactly. Renormalization can be carried out only in the context of perturbation theory. This necessitates the need for a regularization scheme consistent with Fujikawa's formalism. However, Fujikawa's regularization of anomaly is ad-hoc in the sense that only the ill-defined Jacobian factor is regularized in terms of eigenvalues of operator $\not{D}$.There is no prescription for regularizing the Green's functions of the theory. It is, therefore, necessary to first develop a perturbation scheme based on regularization involving eigenvalues of operator $\not{D}$ for calculating Green's functions.

It has been claimed by Verstegen[16] that the regularized Jacobian factor can directly be obtained from a modified Lagrangian density. When the Lagrangian is so modified, the Green's functions of the theory are also regularized in terms of the same cutoff M as appears in the regularized anomaly and one would think that Fujikawa's results are rigourously valid in such a scheme, since all Green's functions are regularized in terms of eigenvalues of the energy operator $\not{D}$. However, the modified Lagrangian density suggested by Verstegen is necessarily of non-polynomial type. The regularization function f(x) in Fujikawa's regularized Jacobian factor is required to satisfy the following properties,

$$f(0)=1 \quad ; \quad f(\infty)=f'(\infty)=\cdots\cdots=0.$$

This precludes f(x) from being a polynomial. Therefore, the usual theorems of renormalization, which apply only to a local

polynomial Lagrangian, do not apply to this modified Lagrangian density, which contains derivatives of fields to arbitrarily high orders and is therefore necessarily non-local.

However, we shall assume that we are working in the context of a regularization scheme in which the Green's functions of the theory are also regularized in terms of eigenvalues of the operator $\not{D}$.

(ii)Fujikawa's procedure leads only to the leading contribution to the trace anomaly[10].However, if this formalism were to yield exact non-perturbative results, one should be able to get exact trace anomaly. This is another indication that Fujikawa's procedure gives only one loop anomaly.

(iii)In the anomaly equation for $\partial_\mu J_5^\mu$, the anomalous term appears as a functional integral

$$\int \mathcal{D}A_\mu\ \mathcal{D}\psi\ \mathcal{D}\bar{\psi}\ A_M(x)\exp[\iota(S_{eff}+\text{source terms})]$$

where $A_M(x)$ is the regularized Jacobian factor and is a functional of gauge fields. $A_M(x)$ is inside the functional integral. As will be shown later, to evaluate $\langle A_M(x)\rangle$, $A_M(x)$ is expanded as an infinite series in the regulator mass M:

$$\begin{aligned} A_M(x) &= \sum_n \phi_n^\dagger(x)\gamma_5\phi_n(x)e^{-\lambda_n^2/M^2} \\ &\equiv O(4)+\frac{O(6)}{M^2}+\frac{O(8)}{M^4}+\cdots\cdots+\frac{O(2n)}{M^{2n-4}}+\cdots \end{aligned} \tag{2.9}$$

where O(2n) turns out to be a gauge invariant functional of dimension 2n. O(6), O(8),...... are themselves of $O(g_0^2)$ (or higher). In evaluating $\langle A_M(x)\rangle$ to $O(g_0^2)$, one needs to take tree Green's functions of O(6), O(8),......which are always finite. Hence to this order

$$\lim_{M^2\to\infty} \langle A_M(x)\rangle = \langle \lim_{M^2\to\infty} A_M(x)\rangle \tag{2.10}$$

which gives the anomaly term. However, as observed also by Shizuya[19,20], this equality may not hold in higher orders. The higher orders terms $O(2n)/M^{2n-4}$, n=3,4···· could contribute in higher orders in perturbation theory. To see this, imagine a perturbation scheme in which Green's functions are also calculated with a cutoff on the eigenvalues of operator $\not{D}$. Then the Green's functions of O(2n) will generally contain divergences of order M^{2n-4} upto factors of $\ln M^2$. [There are no divergences of order M^{2n-2} as will be shown later]. Now when the Green's functions of $O(2n)/M^{2n-4}$ are calculated for finite M^2 and then M^2 is let go to infinity, $\langle O(2n)\rangle/M^{2n-4}$ may contain finite pieces as well as pieces that diverge as $(\ln M^2)^p$. In any case, these terms will lead to non-vanishing and probably divergent contributions in higher orders. Hence, if one is to prove Adler Bardeen theorem, all the terms in the series for $A_M(x)$ should be dealt with. Rigourously, one should first find the divergence structure of $\langle O(2n)\rangle$ and take the limit

$$\lim_{M^2\to\infty} \langle O(2n)\rangle/M^{2n-4}$$

for each n and then sum the series in Eq.(2.9) to obtain the anomaly equation to all orders.

In this chapter, we shall deal with the series in Eq.(2.9) to obtain the following form for the anomaly equation which is valid to all orders,

$$\partial_\mu J^\mu_5 = \frac{2im_o}{1+f(g^2,\ln M^2)}\ \bar\psi\gamma_5\psi - \frac{1}{8\pi^2}\ \frac{1}{1+f(g^2,\ln M^2)}\,\mathrm{Tr}(F_{\mu\nu}\tilde F^{\mu\nu})$$

where $f(g^2, \ln M^2)$ is a power series in $\ln M^2$ and g^2.

To prove the Adler Bardeen theorem in this context, one should prove that the coefficient $f(g^2, \ln M^2)$ is zero. We have not been able to do so entirely within the path integral framework. However, we have succeeded in obtaining the above form for the anomaly equation and then, using this form we have given an arguement for the Adler Bardeen theorem.

[2.2]PRELIMINARIES

In this section, we shall fix our notations and state some results about renormalization of gauge invariant operators, which will be needed in future discussion.

We shall mainly use the notation of Ref.8. γ matrices are as used in Ref.21. γ^0 is hermitian and γ^k(k=1,2,3) are antihermitian, $\gamma^4 \equiv \iota\gamma_0$ is antihermitian. The hermitian γ_5 is defined by $\gamma_5 \equiv \iota\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^1\gamma^2\gamma^3\gamma^4$.

We shall consider the Lagrange density of SU(N) Yang-Mills field coupled to fermions in the fundamental representation of SU(N),

$$\mathcal{L} = \bar{\psi}\, \iota\gamma^\alpha D_\alpha \psi - m_0 \bar{\psi}\psi + \frac{1}{2g^2}\, \mathrm{Tr}(F^{\mu\nu} F_{\mu\nu}) \qquad (2.11)$$

The path integral in Euclidean space is defined by first continuing $\mathcal{L}$ to Euclidean space-time. After the Wick rotation $x^0 \longrightarrow -\iota x^4, A_0 \longrightarrow \iota A_4$ the operator $\not{D} = \gamma_\alpha D^\alpha \equiv \gamma^\alpha(\partial_\alpha + A_\alpha)$ becomes a hermitian operator. Here

$$\iota A_\mu \equiv g A_\mu{}^a(x) T^a$$

$$[T^a, T^b] = \iota f^{abc} T^c \qquad ; \qquad \mathrm{Tr}(T^a T^b) = \tfrac{1}{2}\,\delta^{ab}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \equiv g_o T^a F_{\mu\nu}^{\ a} \tag{2.12}$$

After the Wick rotation the metric becomes

$$g_{\mu\nu} = g^{\mu\nu} = (-1,-1,-1,-1)$$

The functional integral is defined as

$$W[J_\mu, \xi, \bar{\xi}] \equiv \int \mathcal{D}A_\mu \mathcal{D}\psi\ \mathcal{D}\bar{\psi}\ \mathcal{D}c\mathcal{D}\bar{c} \exp\left\{S_{eff}[A,c,\bar{c},\psi,\bar{\psi}] + \int d^4x[J^\mu A_\mu + \bar{\xi}\psi + \bar{\psi}\xi]\right\}$$

$$W[0,0,0] \equiv 1. \tag{2.13}$$

where $S_{eff}[A,c,\bar{c},\psi,\bar{\psi}]$ is the effective action including Faddeev Popov ghost terms of the ghost fields c and $\bar{c}$.

We shall be using the following results regarding renormalization of gauge invariant operators, which have been proved in the context of dimensional regularization[22-24]:

[A]The operators that mix under renormalization with gauge invariant operators are either of the following three types:

(a) Gauge invariant operators

(b) Operators of the form

$$\frac{\delta\tilde{S}}{\delta A_\mu^a}\frac{\delta F}{\delta(\partial_\mu c^a)} + \frac{\delta\tilde{S}}{\delta\psi_a}H_a + \frac{\delta\tilde{S}}{\delta\bar{\psi}_b}\bar{H}_b + \mathcal{S}_o F \tag{2.14}$$

where $\tilde{S}$ = gauge invariant action + ghost action and F is a functional of A_μ^a, $c^a, \bar{c}^a, \psi, \bar{\psi}$ with correct ghost structure[22]. $\mathcal{S}_o$ is the BRS variation operator for $A, c, \psi, \bar{\psi}$.

(c) $$\frac{\delta\tilde{S}}{\delta c^a} X_a[A,c,\bar{c},\psi,\bar{\psi}] \tag{2.15}$$

where X_a is a local functional of its arguments.

[B] The above set of operators is closed under renormalization to all orders.

We shall need only the following two corollaries of the above results,

Corollary 1 : In a Yang-Mills theory with fermions only, there is no operator of dimension two that can mix with a gauge invariant operator to any order.(Proof:There is no such operator amongst the types (a),(b) and (c) listed above).

Corollary 2: The only pseudovector group scalar operator of dimension three (in Yang-Mills theory with fermions only) that can mix with a gauge invariant operator is $\bar{\psi}\gamma_{\mu}\gamma_{5}\psi$.(Proof: This is so because the operators in (b) and (c) have dimension four or more).

The results [A] and [B] have been proved in the context of dimensional regularization. Their validity in case when the regularization is in terms of the cutoff on eigenvalues of $\not{D}$ needs to be verified. However, these results and the corollaries 1 and 2 depend purely on the form of the WT identity of gauge invariant operators which is the same for either regularization, and hence are bound to be valid.

[2.3]PATH INTEGRAL DERIVATION OF CHIRAL ANOMALY

In this section we will briefly review Fujikawa's procedure [7,8] for obtaining the chiral anomaly in the path integral framework and will then explain the one loop nature of his results.

The anomaly equation in path integral formalism is derived as the WT identity obtained by applying the local chiral transformation

$$\psi(x) \longrightarrow \psi'(x) \equiv \exp[i\alpha(x)\gamma_5]\psi(x)$$

$$\bar{\psi}(x) \longrightarrow \bar{\psi}'(x) \equiv \bar{\psi}(x)\exp[\iota\alpha(x)\gamma_5] \tag{2.16}$$

to the functional integral in Eq.(2.13).Under the above transformation, the Lagrangian in Eq.(2.11) changes to

$$\mathcal{L} \longrightarrow \mathcal{L} - \partial_\mu\alpha(x)\bar{\psi}\,\gamma^\mu\gamma_5\psi - 2\iota m_o\alpha(x)\,\bar{\psi}\,\gamma_5\psi \tag{2.17}$$

for an infinitesimal $\alpha(x)$. Fujikawa made the observation that the path integral measure $\mathcal{D}\psi\ \mathcal{D}\bar{\psi}$ is not invariant under chiral transformation. To obtain the variation of measure under chiral transformation he expanded fermionic Grassman variables ψ and $\bar{\psi}$ in terms of eigenfunctions of the operator $\not{D}$. Assuming the system to be enclosed in a large space time box, the eigenvalues of the operator $\not{D}$ are discrete and real. The eigenfunctions satisfy

$$\not{D}\phi_n(x) = \lambda_n\phi_n(x) \tag{2.18}$$

where $\phi_n(x)$ are four spinors satisfying the orthogonality and completeness properties:

$$\int d^4x\ \phi_n^\dagger(x)\phi_m(x) = \delta_{mn} \tag{2.19}$$

and

$$\sum_n \phi_n(x)\phi_n^\dagger(y) = \delta^4(x-y)1 \tag{2.20}$$

One expands ψ and $\bar{\psi}$ as

$$\psi(x) = \sum_n a_n\phi_n(x) \quad ; \quad \bar{\psi}(x) = \sum_n b_n\phi_n^\dagger(x) \tag{2.21}$$

where a_n and b_n are independent Grassman variables.Then the path integral measure is defined as

$$\mathcal{D}\psi\ \mathcal{D}\bar{\psi} = \prod_n da_n \prod_n db_n \tag{2.22}$$

Under the chiral transformation of Eq.(2.16) coefficients a_n and b_n transform as

$$a'_m = \sum_n C_{mn} a_n \quad \text{and} \quad b'_m = \sum_n C_{mn} b_n \tag{2.23}$$

where $C_{mn} = \int d^4x\ \phi^\dagger_m(x)\exp[\iota\alpha(x)\gamma_5]\phi_n(x)$

Taking into account the Grassmanian nature of a_n and b_n we have

$$\prod_m da'_m = (\det C)^{-1}\prod_n da_n \quad ; \quad \prod_m db'_m = (\det C)^{-1}\prod_n db_n \tag{2.24}$$

The Jacobian factor $(\det C)^{-1}$ evaluated for infinitesimal $\alpha(x)$ is

$$(\det C)^{-1} = \exp[-\iota\int d^4x\ \alpha(x)A(x)] \tag{2.25}$$

where,

$$A(x) = \sum_n \phi^\dagger_n(x)\gamma_5\phi_n(x) \tag{2.26}$$

Thus

$$\mathcal{D}\psi'\mathcal{D}\bar{\psi}' = \exp\left\{-2\iota\int d^4x\ \alpha(x)A(x)\right\}\mathcal{D}\psi\ \mathcal{D}\bar{\psi} \tag{2.27}$$

The WT identities are collectively represented by the variational derivative

$$\left[\frac{\delta}{\delta\alpha(x)}\right] W[J_\mu,\xi,\bar{\xi}]\Big|_{\alpha=0} = 0 \tag{2.28}$$

From Eqs.(2.17),(2.27) and (2.28), one obtains the WT identity (in Euclidean space),

$$\langle\partial_\mu J^\mu_5\rangle_{J,\xi,\bar{\xi}} = \langle 2\iota m_0\bar{\psi}\gamma_5\psi\rangle_{J,\xi,\bar{\xi}} + \iota\langle 2A(x)\rangle_{J,\xi,\bar{\xi}} \tag{2.29}$$

where

$$\langle A(x)\rangle_{J,\xi,\bar{\xi}} \equiv \int\mathcal{D}A_\mu\mathcal{D}c\ \mathcal{D}\bar{c}\ \mathcal{D}\psi\ \mathcal{D}\bar{\psi}\ A(x)\exp\ \left\{S_{eff} + \text{source terms}\right\} \tag{2.30}$$

Now $A(x)$ is an ill-defined quantity and to evaluate it Fujikawa introduced a cutoff M:

$$A_M(x) = \sum_n \phi^\dagger_n(x)\gamma_5\phi_n(x)\exp(-\lambda_n^2/M^2) \tag{2.31}$$

As $\phi_n(x)$'s are functionals of A_μ, so is $A_M(x)$. Fujikawa evaluates it by going over to a plane wave basis or

equivalently as follows:

$$\begin{aligned}
A_M(x) &= \sum_n \phi_n^\dagger(x)\gamma_5 \exp(-\lambda_n^2/M^2)\phi_n(x) \\
&= \sum_n \phi_n^\dagger(x)\gamma_5 \exp(-\not{D}^2/M^2)\phi_n(x) \\
&= \lim_{y\to x} \sum_n \phi_n^\dagger(y)\gamma_5 \exp(-\not{D}^2/M^2)\phi_n(x) \\
&= \lim_{y\to x} \mathrm{Tr}\sum_n \gamma_5 \exp(-\not{D}^2/M^2)\phi_n(x)\phi_n^\dagger(y) \\
&= \lim_{y\to x} \mathrm{Tr}\left\{\gamma_5 \exp(-\not{D}^2/M^2)\delta^4(x-y)1\right\} \\
&= \lim_{y\to x} \mathrm{Tr}\left\{\gamma_5 \exp(-\not{D}^2/M^2)\frac{1}{(2\pi)^4}\int d^4k\ \exp[ik.(x-y)]\right\}
\end{aligned} \tag{2.32}$$

where we have used Eq.(2.20). After some simplification one can express $A_M(x)$ as

$$A_M(x) = \mathrm{Tr}\gamma_5\int\frac{1}{(2\pi)^4}\exp(-k_\mu k_\mu/M^2)\exp\left[\frac{D^2+2ik.D+(i/2)\sigma.F}{M^2}\right]$$

Changing the variable $k'_\mu=k_\mu/M$ and dropping the primes one obtains

$$A_M(x) = M^4\,\mathrm{Tr}\gamma_5\int\frac{1}{(2\pi)^4}\exp(-k_\mu k_\mu)\exp\left[\frac{D^2}{M^2}+\frac{2ik.D}{M}+\frac{(i/2)\sigma.F}{M^2}\right] \tag{2.33}$$

Fujikawa evaluates A(x) by taking the limit

$$A(x) = \lim_{M^2\longrightarrow\infty} A_M(x) \tag{2.34}$$

and the result is

$$A(x) = -\frac{1}{16\pi^2}\,\mathrm{Tr}(F_{\mu\nu}\tilde{F}^{\mu\nu}) \tag{2.35}$$

where $\tilde{F}^{\mu\nu}\equiv \frac{1}{2}\,\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$ ($\epsilon^{1234}=1$).

He further shows that the limit in Eq.(2.34) is independent of the regularizing function $f(\lambda_n^2/M^2)$ in Eq.(2.31) provided

$$f(0)=1 \text{ and } f(\infty)=f'(\infty)=\cdots=0 \tag{2.36}$$

Combining Eq.(2.35) with the WT identity of Eq.(2.29) one obtains the well known anomaly equation

$$\langle \partial_\mu J_5^\mu \rangle = \langle 2i m_o \bar{\psi}\gamma_5\psi\rangle - \frac{i}{8\pi^2}\langle \mathrm{Tr}(F_{\mu\nu}\tilde{F}^{\mu\nu})\rangle \tag{2.37}$$

(modulo equations of motion terms)

We shall now give reasons why the above procedure leads to one loop result only. We shall assume that we are working in the context of a regularization scheme consistent with Fujikawa's procedure i.e. a scheme in which the Green's functions as well as the currents are directly regularized in terms of eigenvalues of the operator $\not{D}$. Using such a regularization scheme, the result for the anomaly term in the regularized anomaly equation is a term of the form $\langle A_M(x)\rangle_{J,\xi,\bar{\xi}}$, where

$$\langle A_M(x)\rangle_{J,\xi,\bar{\xi}} = \int \mathcal{D}A_\mu \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, \mathcal{D}c\, \mathcal{D}\bar{c} A_M[A(x)] \exp\left\{S_{eff} + \text{source terms}\right\} \tag{2.38}$$

where the action and hence the Green's functions are regularized in terms of an effective cutoff M.[One could use a different cutoff M' for the gauge field, but it does not change our result as long as M'<M].

As argued in Sec.[2.1], if such a regularization is performed, then the higher order terms in the series of Eq.(2.9) may also have divergences like $M^{2p}(\ln M^2)^q$. A typical term in this series is of the form $\langle O(2n)\rangle_{J,\xi,\bar{\xi}}/M^{2n-4}$ and if $\langle O(2n)\rangle$ has divergences which go worse than M^{2n-4}, then this term does not vanish in the limit $M^2 \longrightarrow \infty$. Thus, as stated

earlier, the interchange of limit $M^2 \longrightarrow \infty$ and the functional integral is valid only in one loop order . As has been remarked by Shizuya[20], the Adler Bardeen theorem will follow only if the path integral measure over A_μ and the $M^2 \longrightarrow \infty$ limit are interchangable, i.e. if

$$\left[\int [dA_\mu], \text{limit } M^2 \longrightarrow \infty\right] (\text{anomalous Jacobian}) = 0 \qquad (2.39)$$

Such an interchange is not valid beyond one loop order. However, we can obtain a general form for the operators O(2n) to get information about their divergence structure. These turn out to be local, gauge invariant operators. Then applying the results stated in Sec.[2.2] for the renormalization of gauge invariant operators, we can obtain a form for the anomaly equation valid to all orders. This is the object of the following sections. After obtaining this form, we shall present an argument for the Adler Bardeen theorem.

[2.4]GENERAL FORM OF O(2n)

As has already been shown, the regularized expression for the anomaly can be expanded as a power series in $1/M^2$:

$$A_M(x) = \sum_n \phi_n^\dagger(x)\gamma_5 \exp(-\lambda_n^2/M^2)\phi_n(x)$$
$$= -\frac{1}{16\pi^2}\mathrm{Tr}(F_{\mu\nu}\tilde{F}^{\mu\nu}) + \frac{O(6)}{M^2} + \frac{O(8)}{M^4} + \cdots + \frac{O(2n)}{M^{2n-4}} + \cdots \qquad (2.9)$$

We will now show that O(2n) can be expressed as

$$O(2n) = \partial^\mu\partial^\nu O^{(1)}_{\mu\nu}(2n-2) + \partial^\mu O^{(2)}_\mu(2n-1) \qquad (2.40)$$

where

$$O^{(2)}_\mu(2n-1) = \left(-\tfrac{1}{4}\right)^{n-2}\frac{1}{(n-2)!}\sum_m \phi_m^\dagger(x)[[\cdots[\gamma_\mu, \not{D}], \not{D}], \cdots]]\gamma_5\phi_m(x)$$

(2.41)

The number of commutators in the above expression is 2n-5. We will show that $O^{(1)}_{\mu\nu}(2n-2)$ and $O^{(2)}_{\mu}(2n-1)$ are local, gauge invariant operators of gauge fields having dimension 2n-2 and 2n-1 respectively.

Here, we will present the proof for the case n=3. The proof for the case n>3, which is along similar lines, is presented in Appendix B. From Eq.(2.9) we see that

$$O(2n) = \frac{1}{(n-2)!} \frac{\partial^{n-2}}{\partial(M^{-2})^{n-2}} A_M(x)\Big|_{M^2\longrightarrow\infty} \tag{2.42}$$

In particular,

$$O(6) = \frac{\partial}{\partial M^{-2}} A_M(x)\Big|_{M^2\longrightarrow\infty} \tag{2.43}$$

and from Eq.(2.9),

$$\frac{\partial}{\partial(M^{-2})} A_M(x) = \sum_m (-\lambda_m^2)\ \phi_m^\dagger(x)\exp(-\lambda_m^2/M^2)\gamma_5\phi_m(x) \tag{2.44}$$

We simplify the expression of Eq.(2.44) by using the following identity(for proof see Appendix A) twice, viz.

$$\phi_m^\dagger(x)X\ \gamma_5\phi_m(x) \equiv \frac{1}{-2\lambda_m}\left\{\phi_m^\dagger(x)[X,\not{D}]\ \gamma_5\phi_m(x) + \partial^\mu[\phi_m^\dagger(x)\gamma_\mu X\ \gamma_5\phi_m(x)\right\} \tag{2.45}$$

provided $\lambda_m \neq 0$.

We obtain, for $\lambda_m \neq 0$,

$$\begin{aligned}\phi_m^\dagger(x)\gamma_5\phi_m(x) &= -\frac{1}{2\lambda_m}\ \partial^\mu[\phi_m^\dagger(x)\gamma_\mu\gamma_5\phi_m(x)] \\ &= \left(-\frac{1}{2\lambda_m}\right)^2\partial^\mu\left\{\phi_m^\dagger(x)[\gamma_\mu,\not{D}]\ \gamma_5\phi_m(x)] + \partial^\nu[\phi_m^\dagger(x)\gamma_\nu\gamma_\mu\gamma_5\phi_m(x)]\right\} \\ &= \frac{1}{4\lambda_m^2}\left\{\partial^2(\phi_m^\dagger(x)\gamma_5\phi_m(x)) + \partial^\mu[\phi_m^\dagger(x)[\gamma_\mu,\not{D}]\ \gamma_5\phi_m(x)]\right\}\end{aligned} \tag{2.46}$$

Thus, using the result of Eq.(2.46) into Eq.(2.44), we obtain

$$\frac{\partial}{\partial M^{-2}} A_M(x) = -\frac{1}{4}\,\partial^2 \sum_{\lambda_m \neq 0} \phi_m^\dagger(x)\gamma_5 \exp(-\lambda_m^2/M^2)\,\phi_m(x)$$
$$-\frac{1}{4}\,\partial^\mu \sum_{\lambda_m \neq 0} \phi_m^\dagger(x)[\gamma_\mu, \not{D}]\gamma_5 \exp(-\lambda_m^2/M^2)\phi_m(x) \tag{2.47}$$

The restriction $\lambda_m \neq 0$ on the summations in Eq.(2.47) can be removed because the additional terms are identically zero:

$$-\partial^2 \sum_{\lambda_m = 0} \phi_m^\dagger(x)\gamma_5\phi_m(x) - \partial^\mu \sum_{\lambda_m = 0} \phi_m^\dagger(x)[\gamma_\mu, \not{D}]\,\gamma_5\phi_m(x) \equiv 0 \tag{2.48}$$

as seen by using $\not{D}\phi_m(x) = 0 = \phi_m^\dagger(x)(\overleftarrow{\not{\partial}} + \not{A})$(for $\lambda_m=0$) in the second term of left hand side of Eq.(2.48). Replacing $\exp(-\lambda_m^2/M^2)\phi_m(x)$ by $\exp(-\not{D}^2/M^2)\phi_m(x)$ we obtain,

$$O(6) = \partial^\mu\partial^\nu O^{(1)}_{\mu\nu}(4) + \partial^\mu O^{(2)}_\mu(5) \equiv \partial^\mu\partial^\nu [g_{\mu\nu} O^{(1)}(4)] + \partial^\mu O^{(2)}_\mu(5) \tag{2.49}$$

where

$$O^{(1)}(4) = \lim_{M^2\to\infty} -\frac{1}{4} \sum_m \phi_m^\dagger \gamma_5 \exp(-\not{D}^2/M^2)\phi_m \tag{2.50}$$

$$O^{(2)}_\mu(5) = \lim_{M^2\to\infty} -\frac{1}{4} \sum_m \phi_m^\dagger [\gamma_\mu, \not{D}]\gamma_5 \exp(-\not{D}^2/M^2)\phi_m \tag{2.51}$$

$O^{(1)}(4)$ and $O^{(2)}_\mu(5)$ can be calculated as was done by Fujikawa[8] and are clearly local operators of gauge fields as seen by direct calculation[see Appendix B;Eq.(B.14)]. They are also gauge invariant operators because as shown in Appendix B any operator of the form

$$\lim_{M^2\to\infty} \sum_m \phi_m^\dagger f(\not{D}) \exp(-\not{D}^2/M^2)\phi_m$$

where $f(\not{D})$ is a matrix operator function of $\not{D}$ is gauge invariant. $O^{(1)}(4)$ and $O^{(2)}_\mu(5)$ have dimension 4 and 5 respectively. The result for $O^{(1)}(4)$ and $O^{(2)}_\mu(5)$ is as follows:

$$O^{(1)}(4) = \frac{1}{64\pi^2} \quad Tr(F_{\mu\nu}\tilde{F}^{\mu\nu})$$

$$O^{(2)}_{\mu}(5) = \frac{1}{96\pi^2} \left[-\tfrac{1}{2}\partial_{\mu}(F\tilde{F}) + \tfrac{1}{2}D^{ab\eta}F^{b}_{\eta\delta}\tilde{F}^{a\delta}_{\mu}\right]$$

$$\equiv \frac{1}{96\pi^2} \left[-\tfrac{1}{2}\partial_{\mu}(F\tilde{F}) + \tilde{O}^{(2)}_{\mu}(5)\right] \tag{2.52}$$

Eq.(2.40) can be proved in a similar manner as we have shown in Appendix B. The form of operator $O^{(1)}_{\mu\nu}(n-2)$ is not needed as it will be shown later that it does not contribute to the anomaly equation.

[2.5]GENERAL FORM OF HIGHER ORDER CORRECTIONS

We shall now derive the general form of higher order contributions coming from the Jacobian using the general form of O(2n) established in Sec.[2.3].To achieve this end, we shall make use of the corollaries 1 and 2 stated in Sec.[2.2] about the renormalization of gauge invariant operators.

We shall assume that the Green's functions of operators have been suitably regularized using cutoff M on the large eigenvalues of of the operator $\not{D}$. (Our results are of general nature of the form of higher order contributions and hence probably independent of the details of regularization). The Green's functions will contain divergences of the kind $M^{2p}(\ln M^2)^q$, where p is restricted from above by power counting. Consider now Green's functions of O(2n) whose form is given in Eqs.(2.40) and (2.41). In Eq.(2.29), O(2n) appears in the form $\langle O(2n)\rangle/M^{2n-4}$. Hence in the limit $M^2 \longrightarrow \infty$, only that part of $\langle O(2n)\rangle$ which diverges as M^{2n-4} or worse could possibly

contribute to $\lim_{M^2\to\infty} \langle A_M(x)\rangle$. It is clear that the contribution to $\langle A_M(x)\rangle$ from higher order terms comes entirely from divergences in Green's functions $\langle O(2n)\rangle$ and these too of sufficiently high order.

Now, $O(2n)$ is a linear combination of terms that involve two local, gauge invariant operators $O^{(1)}_{\mu\nu}$ and $O^{(2)}_{\mu}$. Consider first the divergence structure of $O^{(1)}_{\mu\nu}(2n-2)$. We need to worry only about the divergences in $\langle O^{(1)}_{\mu\nu}(2n-2)\rangle$ that go like M^{2n-4} or worse. By counting dimensions such divergences are necessarily of the form ,

$$M^{2n-4}(\ln M^2)^q \times \text{ local operator of dimension two.}$$

But $O^{(1)}_{\mu\nu}(2n-2)$ is a gauge invariant operator as shown in Appendix B. Therefore, from corollary 1 of Sec.[2.2], there is no gauge invariant operator of dimension two it can mix with. Hence, such terms must be absent:

$$\lim_{M^2\to\infty} \langle O^{(1)}_{\mu\nu}(2n-2)\rangle/M^{2n-4} = 0 \qquad (2.53)$$

By a similar logic, the divergences in $\langle O^{(2)}_{\mu}(2n-1)\rangle$ one needs to worry about are of the form,

$$M^{2n-4}(\ln M^2)^q \times \text{ local pseudovector operator of dimension three.}$$

By corollary 2 of Sec.[2.2], the only such operator is $\bar{\psi}\gamma_\mu\gamma_5\psi$

Hence, as $M^2\to\infty$,

$$\frac{\langle(O^{(2)}_{\mu}(2n-1)\rangle}{M^{2n-4}} = \frac{1}{2}\, f_n(g^2, \ln M^2)\, \bar{\psi}\gamma_\mu\gamma_5\psi + O(1/M^2) \qquad (2.54)$$

where $f_n(g^2, \ln M^2)$ is a power series in g^2 and $\ln M^2$. Combining Eqs.(2.53) and (2.54) with Eq.(2.9), we obtain

$$\lim_{M^2\to\infty} \langle A_M(x)\rangle = -\frac{1}{16\pi^2} \langle \mathrm{Tr}(F_{\mu\nu}\tilde{F}^{\mu\nu})\rangle + \frac{1}{2}\, f(g^2, \ln M^2)\partial^\mu \langle\bar{\psi}\gamma_\mu\gamma_5\psi\rangle$$

(2.55)

where

$$f(g^2,\ln M^2)=\sum_{n=3}^{\infty} f_n(g^2,\ln M^2)$$

Substituting the above in Eq.(2.29), we obtain the following form for the anomaly equation (in Minkowski space) valid to all orders:

$$\langle\partial_\mu J^\mu_5\rangle = \frac{2i m_0}{1+f(g^2,\ln M^2)} \langle\bar{\psi}\gamma_5\psi\rangle - \frac{1}{8\pi^2}\frac{1}{1+f(g^2,\ln M^2)}\langle Tr(F_{\mu\nu}\tilde{F}^{\mu\nu})\rangle \tag{2.56}$$

[2.6]ADLER BARDEEN THEOREM

It is clear from Eq.(2.56) that proving Adler Bardeen theorem amounts to showing that,

$$f(g^2,\ln M^2)=0$$

A direct calculation shows that $f_n(g^2,\ln M^2)$ is a power series in g^2 beginning as g^4. Further, all the operators $O(2n)$, $n \geq 3$ will generally contribute to a given order $(g^2)^m$ in $f(g^2,\ln M^2)$. This makes the evaluation of $f(g^2,\ln M^2)$ technically difficult. However, one can give the following indirect arguement for the Adler Bardeen theorem.

As shown by Fujikawa[8], the usual Pauli Villars treatment is an ideal perturbative realization of the path integral formulation. In this formulation the anomaly comes not from the Jacobian(these cancel between the regulator field and the fermion field measures), but from the regulator field mass term. In the standard perturbative treatment it is well known

that the coefficient of the naive divergence term $2\iota m_o\bar{\psi}\gamma_5\psi$ is not modified to all orders. This follows very simply from an elementary diagrammatic considerations. If we borrow just this much from the perturbative approach, then Eq.(2.56) implies that

$$f(g^2, \ln M^2) = 0$$

This in turn proves the Adler Bardeen theorem to all orders:

$$\partial_\mu J_5^\mu = 2\iota m_o\langle\bar{\psi}\gamma_5\psi\rangle - \frac{1}{8\pi^2}\langle Tr(F_{\mu\nu}\tilde{F}^{\mu\nu})\rangle \tag{2.57}$$

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CHAPTER 3

RENORMALIZATION OF ENERGY MOMENTUM TENSOR AND FINITE IMPROVEMENT PROGRAM

[3.1]INTRODUCTION

Given a law of physics in flat space-time, the principle of general covariance dictates how to extend this law to curved space-time. In general, given the action $S[\phi]$ in flat space, the way to include the effect of gravitation, is to make this action generally covariant. The minimal way to obtain $S[\phi,g]$, the action in presence of gravitation, is to replace $\eta_{\mu\nu}$ by $g_{\mu\nu}$ and ordinary derivatives by covariant derivatives in $S[\phi]$ e.g. the flat space action of $\lambda\phi^4$-theory

$$S[\phi] = \int d^4x[\ \tfrac{1}{2}\ \partial_\mu\phi\partial^\mu\phi\ -\ \tfrac{1}{2}m_0^2\phi^2\ -\ \frac{\lambda_0}{4!}\phi^4]$$

is written in minimal generally covariant form as

$$S[\phi,g] = \int d^4x\sqrt{-g}(x)[\ \tfrac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\ -\ \tfrac{1}{2}m_0^2\phi^2\ -\ \frac{\lambda_0}{4!}\phi^4]$$

The resulting theory is called minimal Einstein theory. [However, general Einstein theories, involving a direct coupling of the curvature tensor with the matter field variables are not excluded by experimental evidence].

The energy momentum tensor, which is the source of gravitation is obtained from $S[\phi,g]$ via

$$\theta_{\mu\nu} = \frac{2}{\sqrt{-g}}\ \frac{\delta S}{\delta g^{\mu\nu}}\bigg|_{g^{\mu\nu}=\eta^{\mu\nu}}$$

This $\theta_{\mu\nu}$ is needed to describe scattering in a weak external gravitational field. Its matrix elements describe a process in which gravitational waves are emitted and hence these matrix elements are observables. It is therefore desirable that $\theta_{\mu\nu}$ be finite. For this reason, the renormalization of energy momentum tensor has great importance in any quantum field theory and it has been studied extensively[1-8].

In general, the energy momentum tensor obtained from the minimal action is not finite even in one loop order but there is always the possibility of obtaining a finite $\theta_{\mu\nu}$ to all orders from a non minimal generally covariant action. For example, the conformally invariant action

$$S[\phi,g] = \int d^4x\sqrt{-g}\,[\mathcal{L}_{min} - \frac{1}{12} R\phi^2]$$

leads to an improved energy momentum tensor

$$\bar{\theta}_{\mu\nu} = \theta_{\mu\nu} - \frac{1}{6}(\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2)\phi^2$$

which has finite matrix elements to one loop order [2]. One may expect to obtain a more general coefficient of the improvement term which will give a finite $\theta^{imp}_{\mu\nu}$ to all orders. However, how one renormalizes the energy momentum tensor is also an important question. It is generally hoped that the parameters of the flat space theory be sufficient to fix the theory in curved space[4]. If the energy momentum tensor is renormalized by introducing new infinite counterterms apart from those of the flat space theory, then one has to put additional renormalization conditions to determine these counterterms and in the process one has to introduce new parameters which have

to be specified by the experiment. Therefore one would like to renormalize $\theta_{\mu\nu}$ without requiring any additional infinite counterterms.

Another reason why one studies the renormalization of energy momentum tensor is that it appears in the operator product expansion (O.P.E) of two currents[9]. This O.P.E. is related to physical scattering matrix elements and therefore the finiteness (or the lack of finiteness) of the energy momentum tensor and its anamolous dimension have observable consequences.

Renormalization of energy momentum tensor in quantum field theories has been a subject of keen interest. Freedman, Muzinich and Weinberg [3] have studied the energy momentum tensor in gauge theories in the context of theory without scalars and they have shown that its matrix elements are finite and gauge independent. In theories with scalar fields it is well known that the energy momentum tensor obtained from

$$\theta_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}}\bigg|_{g^{\mu\nu}=\eta^{\mu\nu}}$$

does not lead to a finite energy momentum tensor[2]. In the context of $\lambda\phi^4$-theory it was shown first by Callan, Coleman and Jackiw [2] that one can improve the conventional energy momentum tensor by adding to it certain terms in such a manner that the improved energy momentum tensor is finite to one loop order [Such an improvement term is, of course, admissible provided it does not alter the energy and momentum of the system]. Since then, the idea of improving the energy momentum tensor of scalar

theories so as to render its matrix elements finite, has been pursued by various authors [1-8].

An important question involved in the construction of such an improved, finite energy momentum tensor is whether in renormalizing it an extra renormalization condition is needed or not [4]. There may be choices of improvement coefficients that may not allow extra renormalization condition. Such a choice is the most desirable one. As will be shown in later chapters, there is a definite choice of the form of operator only which can be added as an improvement term (for example, in case of a real scalar field the only improvement term that can be added is proportional to $(\partial^2 g_{\mu\nu} - \partial_\mu \partial_\nu)\phi^2$. The crucial point that has to be settled is the choice of a specific form for the proportionality factor for this term. In the framework of dimensional regularization, this coefficient H_o can be chosen either as a function of ε (=4-n, where n is the space - time dimension), bare coupling constants and bare masses or as a function of ε, renormalized coupling constants and renormalized masses. In either of these situations H_o should be a finite function of its arguments. The reason why one requires H_o to be a finite function of its arguments is as follows:

If one requires the improved energy momentum tensor to be directly obtainable from an action, then one needs to modify the action also by adding to it an extra term $\frac{1}{2}\varkappa_o R\phi^2$. Now if the improvement coefficient H_o is not a finite function of its arguments, then an independent renormalization of $\varkappa_o$ is needed

and therefore an additional piece of experimental information is needed to fix the theory, this information being the "root mean square mass radius" of the scalar particle [4]. Thus, to renormalize the energy momentum tensor without requiring any new parameter one needs a finite improvement program.

But the finite improvement program can still be interpreted in two different ways. One in which the improvement coefficient is a finite function of bare parameters and the other in which it is a finite function of the renormalized parameters of the theory. In past,various authors have tried to obtain finite energy momentum tensors in $\lambda\phi^4$-theory choosing the improvement coefficient as a finite function of renormalized parameters[6-8]. We emphasize on the desirability of the former choice which is more gerneral also[10]. We shall stress on this point in Section (3.3). In the following chapters, we shall consider $\lambda\phi^4$- theory as well as theories involving scalar fields and two coupling constants i.e. scalar electrodynamics, non-abelian gauge theories with scalars, Yukawa theory and a theory with two interacting scalar fields. In the context of $\lambda\phi^4$-theory,we shall prove that it is possible to obtain a unique finite energy momentum tensor which is a finite function of bare quantities.This is the same as the energy momentum tensor obtained by Collins [6]. In other four theories we shall prove that it is not possible to improve the energy momentum tensor in either of the two ways stated above[10-14].

In Section (3.2), we shall review the work of previous authors in order to prepare the necessary background for our work. In Section (3.3), we shall discuss the various interpretations of finite improvement program and shall make the difference between them clear. We shall then, emphasize on the desirability of having a finite improvement program in which the improvement coefficient is a finite function of bare quantities.

[3.2] REVIEW OF PREVIOUS WORK

Callan, Coleman and Jackiw [2] were the first to study the finiteness of energy momentum tensor in $\lambda\phi^4$-theory. The minimal Einstein tensor defined by

$$\theta_{\mu\nu} = \frac{2}{\sqrt{-g}} \left. \frac{\delta S}{\delta g^{\mu\nu}(x)} \right|_{g^{\mu\nu}(x)=\eta^{\mu\nu}} \tag{3.1}$$

reduces for scalar theory with a ϕ^4-interaction to

$$\theta_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathscr{L} \tag{3.2}$$

where

$$\mathscr{L} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m_o^2\phi^2 - \frac{\lambda_o}{4!}\phi^4 \tag{3.3}$$

But this $\theta_{\mu\nu}$ is not finite even to one loop order . However, one can add to $\theta_{\mu\nu}$ an additional term so as to make it finite <u>provided</u> the new term does not affect the conserved quantities $\int\theta^c_{\mu o}(x)\, d^3x$. The only term satisfying these conditions is proportional to

$$(\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2)\phi^2 \tag{3.4}$$

Callan, Coleman and Jackiw modified the energy momentum tensor

of Eq.(3.2) by adding such an improvement term (called the CCJ improvement term) to obtain

$$\bar{\theta}_{\mu\nu} = \theta_{\mu\nu} + H_0 (\partial_\mu \partial_\nu - g_{\mu\nu} \partial^2) \phi^2 \tag{3.5}$$

This improved energy momentum tensor can be derived from a modified action

$$S = \int d^4x \sqrt{-g(x)} (\mathcal{L} - \tfrac{1}{2} H_0 R \phi^2) \tag{3.6}$$

The non-minimal term $-\frac{1}{2} H_0 R\phi^2$ vanishes in the flat space - time limit but its functional derivative with respect to $g_{\mu\nu}$ does not and it actually gives the improvement term in $\bar{\theta}_{\mu\nu}$. CCJ proved that this energy momentum tensor is finite to $O(\lambda)$ if one chooses $H_0 = -\frac{1}{6}$. Freedman, Muzinich and Weinberg [3] studied the finiteness of energy momentum tensor with CCJ improvement term beyond one loop order. They showed that even the CCJ improved energy momentum tensor is not finite beyond one loop order. They used the Ward and Trace identities to establish the following results in the context of $\lambda\phi^4$- theory and the non abelian gauge theories (with or without scalars).

[A] The energy momentum tensor has finite matrix elements at zero external momentum q and to first order in q as a consequence of Ward identity.

[B] It is suffucient to establish the finiteness of trace of $\theta_{\mu\nu}$ in order to establish the finiteness of $\theta_{\mu\nu}$.

We shall make use of these two results in the following chapters.Their proof for $\lambda\phi^4$-theory and NAGTs with scalars has already been given by Freedman, Muzinich and Weinberg[3]. Still, for the sake of completeness, we prove below these two

statements sketchily in the context of $\lambda\phi^4$-theory.

The Ward and Trace identities of $\lambda\phi^4$-theory are

$$q^{\mu}\Gamma^{(j)}_{\mu\nu}(q;p_1,\cdots\cdots p_j) = -\iota\sum_{\iota}(q+p_{\iota})_{\nu}G(p_1,\cdots\cdots p_{\iota-1},p_{\iota}+q,\cdots p_j) \quad (3.7)$$

and

$$g_{\mu\nu}\Gamma^{(j)}_{\mu\nu}(q;p_1,\cdots p_j) = \Gamma^{(j)}(q;p_1,\cdots p_j) - \iota\sum_{\iota}G(p_1\cdots p_{\iota}+q,\cdots p_j) \quad (3.8)$$

respectively, where

$$G^{(j)}(p_1,\cdots p_j) = \text{F.T.}\left\{\iota^{-(j-1)}\frac{\delta^j}{\delta J^R(x_1),\cdots\cdots\cdots\delta J^R(x_j)}\ln W^R[J]\right\}$$

$$\Gamma^{(j)}_{\mu\nu}(q;p_1,\cdots p_j) =$$

$$\text{F.T.}\left\{\iota^{-j}\frac{\delta^j}{\delta J(x_1),\cdots\cdots\cdots\delta J(x_j)}\frac{1}{W[J]}\int d\phi\ \theta_{\mu\nu}(y)\exp(\iota\textstyle\int d^4x[\mathcal{L}+J\phi]\right.$$

$$\Gamma^{(j)}(q;p_1,\cdots p_j) =$$

$$\text{F.T.}\left\{\iota^{-j}\frac{\delta^j}{\delta J(x_1),\cdots\cdots\cdots\delta J(x_j)W[J]}\int d\phi\ \theta_{\mu}^{\ \mu}(y)\exp(\iota\textstyle\int d^4x[\mathcal{L}+J\phi])\right.$$

$$(3.9)$$

Differentiating Eq.(3.7) w.r.t q and noting that the $G^{(j)}$'s are finite (because the theory is finite) one obtains

$$\Gamma^{(j)}_{\mu\nu}(q;p_1,\cdots\cdots,p_j)\Big|_{q_\alpha = 0} = \text{finite} \quad (3.10)$$

and

$$\frac{d}{dq^{\alpha}}\Gamma^{(j)}_{\mu\nu}(q;p_1,\cdots\cdots,p_j)\Big|_{q_\alpha = 0} = \text{finite} \quad (3.11)$$

Eq.(3.10) and Eq.(3.11) are nothing but the mathematical forms of statement [A]. To prove statement [B] we proceed as follows:

Since $\theta_{\mu\nu}$ is a dimension 4 operator only $\Gamma^{(j)}(q;p_1,\cdots\cdots,p_j)$ with j=1,2,3,4 have possibly divergent Taylor series coefficients. $\Gamma^{(1)}_{\mu\nu}$ and $\Gamma^{(3)}_{\mu\nu}$ occur only if the discrete symmetry

$\phi(x) \longrightarrow -\phi(x)$ of the ϕ^4 Lagrangian is spontaneously broken. The Taylor series may be written as follows with finite terms of order higher than 4-j being ignored,

$$\Gamma^{(4)}_{\mu\nu}(q;p_1,\cdots,p_4) = a_1 g_{\mu\nu} + \cdots$$
$$\Gamma^{(3)}_{\mu\nu}(q;p_1,p_2,p_3) = a_2 g_{\mu\nu} + \cdots$$
$$\Gamma^{(2)}_{\mu\nu}(q;-p,p) = [a_3 + a_4 q^2 + a_5 (p+p')^2] g_{\mu\nu} + a_6 (p+p')_\mu (p+p')_\nu + a_7 [q_\mu (p+p')_\nu + (p+p')_\mu q_\nu] + a_8 [q_\mu q_\nu - g_{\mu\nu} q^2]$$
$$\Gamma^{(1)}_{\mu\nu}(q;-q) = a_9 g_{\mu\nu} + a_{10} g_{\mu\nu} q^2 + a_{11} [q_\mu q_\nu - g_{\mu\nu} q^2] \qquad (3.12)$$

Eqs. (3.10) and (3.11) then imply that a_1, a_2, a_3, a_5, a_6, a_7 and a_9 are finite. Then applying the W.T.identity (3.7) one can immediately see that a_4 and a_{11} are also finite. Now to $O(\lambda)$, the trace identity (3.8) holds which implies that a_8 and a_{11} are also finite. This completes the proof of finiteness of $\theta_{\mu\nu}$ to $O(\lambda)$. If one goes beyond one loop order there may be divergences in $\Gamma^{(j)}(q;p_1,\cdots,p_j)$ of the form $(q_\mu q_\nu - g_{\mu\nu} q^2)$. If one knows that $\Gamma^{(j)}(q;p_1,\cdots p_j)$ were finite to all orders, then the same result would be obtained for $\Gamma^{(j)}_{\mu\nu}$ by similar method as applied above. Hence it is sufficient to prove the finiteness of $\theta_\mu{}^\mu$ to prove the finiteness of $\theta_{\mu\nu}$.

Making use of the results [A] and [B] Freedman and Weinberg [4] considered the problem of finiteness of $\theta_{\mu\nu}$ in the context of dimensional regularization and showed that by modifying $\theta_{\mu\nu}$ to

$$\theta^{imp}_{\mu\nu} = -g_{\mu\nu}\mathcal{L} + \partial_\mu\phi\,\partial_\nu\phi + H_0(\lambda)(\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2)\phi^2$$

where $H_0(\lambda)$ is an appropriately chosen finite function of the renormalized coupling constant, a finite energy momentum tensor

upto $O(\lambda^2)$ can be obtained. This was called by them the finite improvement program. But even this finite improvement program failed beyond three-loops, i.e. for no finite choice of $H_o(\lambda)$ one could obtain a finite $\theta^{imp}_{\mu\nu}$.

Collins [6] treated the problem from many angles and suggested H_o, if possible, as a function, finite at n=4, of ε(= 4-n) and renormalized coupling and mass, so that $H_o(\lambda,\varepsilon)$ reduces to $-\frac{1}{6}$ in the limit n=4 and λ=0. He then succeeded in proving that such an energy momentum tensor does exist, is unique and is a finite function of ε only. The improvement coefficient $H_o(\varepsilon)$ was a power series in ε and the coefficients of ε^r could be chosen appropriately so as to render the matrix elements of $\theta_{\mu\nu}$ finite to all orders in perturbation theory.

[3.3]FINITE IMPROVEMENT PROGRAM

In $\lambda\phi^4$-theory as well as in all theories involving scalar fields and having two coupling constants the energy momentum tensor obtained from Eq.(3.1),where $S[\phi,g]$ is the minimal generally covariant action (i.e. one obtained by making the flat space action generally covariant, without adding any new term) is found to have divergent matrix elements. In $\lambda\phi^4$-theory, the CCJ improved energy momentum tensor is finite only to one loop order. Even the energy momentum tensor

$$\theta^c_{\mu\nu} = \theta_{\mu\nu} - \frac{n-2}{4(n-1)}(\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2)\phi^2 \tag{3.13}$$

which is derived from a conformally invariant action[7,8]

$$S = S_o - \frac{n-2}{8(n-1)}(\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2)\phi^2 \tag{3.14}$$

does not have finite matrix elements beyond three loops. However one may still add to this any quantity whose divergence is identically zero and which does not contribute to Ward identities. In fact, a non zero improvement term is necessarily needed to make the Green's functions of $\theta_{\mu\nu}$ finite. The only such term we need to consider is proportional to $(\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2)\phi^2$. Thus there is a possibility of renormalizing $\theta_{\mu\nu}$ by using the finite improvement program. But the finite improvement program can be interpreted in different ways. The simplest (although unsuccessful) interpretation by Freedman and Weinberg considered the improvement coefficient to be a function of renormalized coupling constant only

$$\theta^{FW}_{\mu\nu} = \bar{\theta}_{\mu\nu} + H_o(\lambda)\ (\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2)\phi^2 \tag{3.15}$$

where $H_o(\lambda)$ is a power series in λ. But this finite improvement program failed beyond $O(\lambda^3)$ as it was not possible to make this energy momentum tensor finite without introducing infinite counterterms beyond those required to renormalize Green's functions of the flat space theory. These counterterms are introduced by modifying the improvement term in the Lagrangian as[4]

$$\frac{1}{12} R\phi^2 \longrightarrow \frac{1}{12} Z_g (1+g) R\phi^2 \tag{3.16}$$

Z_g is to be determined order by order so that the renormalization condition

$$\frac{d}{dq^2}[\ \Gamma^{(2)}(q;p,\ -p\ -q)]\Big|_{q^2=0} = -g \tag{3.17}$$

is satisfied.

There is no particular reason to choose a preferred value

of g and it has to be determined experimentally.

Collins chose a more general improvement coefficient i.e., one which is a finite function of λ, m^2 and ε[6],

$$\theta'_{\mu\nu} = \bar{\theta}_{\mu\nu} - H_o(\lambda, m^2/\mu^2, \varepsilon)(\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2]\phi^2 \tag{3.18}$$

He then showed that there is a unique $H_o(\lambda, m^2/\mu^2, \varepsilon)$ which gives finite matrix elements for $\theta'_{\mu\nu}$ to all orders in perturbation theory and that this improvement coefficient is a finite function of ε only.

The point we wish to make here is that there is a more desirable choice of H_o than the one chosen by Collins and other authors [6-8]. The CCJ improved energy momentum tensor of Eq.(3.13) is derivable from the action in Eq. (3.14). Now if we want the energy momentum tensor to be derivable from an action that is a finite function of bare quantities, then the improvement coefficient H_o should be a finite function of only the bare parameters of the theory. Therefore, a more appropriate improved energy momentum tensor would be of the form

$$\theta''_{\mu\nu} = \bar{\theta}_{\mu\nu} + H_o(\lambda_o\mu^{-\varepsilon}, m_o^2/\mu^2, \varepsilon)(\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2)\phi^2 \tag{3.19}$$

This energy momentum tensor is derivable from an action, which is a finite function of bare quantities:

$$S'' = S - \tfrac{1}{2}H_o(\lambda_o\mu^{-\varepsilon}, m_o^2/\mu^2, \varepsilon)R\phi^2 \tag{3.20}$$

and no new renormalization conditions are needed at all. On the other hand, energy momentum tensor in Eq. (3.18) is not derivable from such a bare action. The improvement term, in this case, can also be derived from an additional term in

action, the full action being

$$S' = S + \varkappa_o R\phi^2 \tag{3.21}$$

where $\varkappa_o$ is now a finite function of renormalized quantities. It can also be written as

$$\varkappa_o(\lambda, m^2/\mu^2, \varepsilon) R\phi^2 = \varkappa_o(\lambda_o \mu^{-\varepsilon}, m_o^2/\mu^2, \varepsilon) + Z_{\varkappa_o} \tag{3.22}$$

where $\varkappa_o$ is a finite function of bare parameters also and $Z_{\varkappa_o}$ are infinite counterterms needed to make $\varkappa_o$ a finite function of renormalized parameters. Thus, there is a need to add additional infinite counterterms to the theory but these counterterms are simply determined by requiring the right hand side of Eq.(3.22) to be finite. Thus, we have a theoretical reason (i.e. requirement of finiteness of H_o) to fix a particular value of g.

To summarize, there are two distinct ways of improving the energy momentum tensor without needing any new information from the experiment:

(a) One, in which the improvement coefficient is a finite function of bare quantities. In this case, the improved energy momentum tensor is derivable from a bare action, as it should be and the coefficient of the non-minimal term $R\phi^2$ in the action is not renormalized. This one may lead to finite matrix elements (to first order in $h_{\mu\nu}$) for the process $A \longrightarrow B$ in which gravitational radiation is emitted and no new parameters are needed.

(b) One, in which the improvement coefficient is a finite

function of renormalized parameters. In this case, the improved energy momentum tensor is not derivable from a finite bare action. But the coefficient $\varkappa_o$ of the $\varkappa_o R\phi^2$ term need not be independently renormalized.

Both of these forms can lead to a finite energy momentum tensor but the choice (a) is more desirable than the choice (b) because this form is derivable from a finite bare action.

However, at least in case of $\lambda\phi^4$- theory, both of these finite improvement programs lead to one and the same energy momentum tensor (which is unique and is a function of ε only). As shown by Collins, if one chooses an improvement term of Eq. (3.18), then one can obtain a <u>unique</u> finite energy momentum tensor and the corresponding H_o is a function of ε only. In chapter 4, we shall consider the possibility of having an improvement term of the form in Eq. (3.19) and will prove that if an energy momentum tensor of the form in Eq. (3.19) exists and is finite, then it is unique and hence coincides with the energy momentum tensor obtained by Collins.

In chapters 5-7, we shall investigate the possibility of obtaining finite energy momentum tensors, using finite improvement program, in theories involving scalar fields and having two coupling constants. We shall consider four such theories:

(a) Scalar Quantum Electrodynamics

(b) Non-abelian gauge theory with scalars

(c) Yukawa theory

(d) A theory with two interacting scalar fields.

We shall consider finite improvement program of type (a) as well as type (b) and then establish, for each of these four cases, the impossibility of obtaining a finite energy momentum tensor without introducing additional infinite counterterms.

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CHAPTER 4

A UNIQUENESS THEOREM REGARDING $\theta_{\mu\nu}$ IN SCALAR THEORY

[4.1] INTRODUCTION

In scalar ϕ^4-theory with Lagrange density expressed in terms of bare quantities

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\,m_o^2\,\phi^2 - \frac{\lambda_o}{4!}\,\phi^4 \tag{4.1}$$

one has the canonical energy momentum tensor

$$\theta^C_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L} \tag{4.2}$$

Callan, Coleman and Jackiw [1] showed that the above energy momentum tensor does not have finite matrix elements to one loop order. It can be shown that the most general modification of $\theta^C_{\mu\nu}$ that does not affect the conserved quantities $\int\theta^C_{\mu o}(x)d^3x$ is

$$\theta_{\mu\nu} = \theta^C_{\mu\nu} + H_o(\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2)\phi^2 \tag{4.3}$$

Collins [2] showed that with $H_o = H_o(\varepsilon)$, a unique finite function of ε only, one obtains an energy momentum tensor, which is finite to <u>all</u> orders in perturbation theory.

In scalar theory, there are an infinite number of energy momentum tensors [3] of the form

$$\theta_{\mu\nu} = \theta^C_{\mu\nu} + H_o(\lambda, m^2/\mu^2, \varepsilon)(\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2)\phi^2 \tag{4.4}$$

that have finite matrix elements to all orders [Here μ is an arbitrary mass scale introduced in dimensional regularization].

Energy momentum tensors can be obtained from actions via

either the canonical procedure or as $- \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} \Big|_{g^{\mu\nu}=\eta^{\mu\nu}}$, where S is the action of the fields coupled to external background gravitational field $g_{\mu\nu}$. Total actions (including renormalization counterterms) are generally finite fuctions of bare quantities. Consequently, the energy momentum tensors derived from them via either of the two procedures are finite functions of bare quantities (for example, the energy momentum tensor in gauge theories is a finite function of bare quantities[4]).

It is, therefore, natural to enquire whether there is an energy momentum tensor in scalar theories that is a finite function of bare quantities and which has finite matrix elements to all orders in perturbation theory and whether it is a unique energy momentum tensor. Such an energy momentum tensor

$$\theta_{\mu\nu} = \theta^{c}_{\mu\nu} + H_o(\lambda_o\mu^{-\varepsilon}, m_o^2/\mu^2, \varepsilon)(\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2)\phi^2 \qquad (4.5)$$

where H_o is a finite function of its arguments $\lambda_o\mu^{-\varepsilon}$ and m_o^2/μ^2 (at $\varepsilon=0$), can then be derived from an action [5]

$$S[\phi, g] = \int \mathcal{L} d^n x + \frac{1}{2} H_o(\lambda_o\mu^{-\varepsilon}, m_o^2/\mu^2, \varepsilon) \int R\phi^2 d^n x \qquad (4.6)$$

which, itself is a finite function of bare parameters of flat space theory only.

Any other energy momentum tensor except the ones in Eqs. (4.4), where H_o is a finite function of λ and m, and (4.5) can be derived from an action that is a finite function of bare quantities only with the introduction of a new coupling constant for the $\frac{1}{2}R\phi^2$ term and this coupling constant has to be

renormalized independently.

The answer to the first question about the existence of such an energy momentum tensor is, of course, in affirmative. The energy momentum tensor constructed by Collins [2],

$$\bar{\theta}_{\mu\nu} = \theta^{C}_{\mu\nu} + H_o(\varepsilon)(\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2)\,\phi^2 \tag{4.7}$$

is of the form in Eq (4.5) and is finite. In this chapter, we will show that it is unique finite energy momentum tensor of the form in Eq. (4.5) also.

The result obtained here differs from the uniqueness theorem of Collins in that Collins considered uniqueness of energy momentum tensor of the form of Eq.(4.4) with H_o a finite quantity (i.e. a finite function of renormalized quantities). His H_o is then not a finite function of bare quantities except in the special case when it is a function of ε only.

In Sec.(4.2), we shall fix our notations. In Sec.(4.3), we shall obtain the most general form of $\theta_{\mu\nu}$ and show that it is indeed of the form in Eq.(4.5). In Sec.(4.4), we will show that the energy momentum tensor of Eq. (4.5) which is finite to all orders is a unique energy momentum tensor. In Sec.(4.5), we will show that this energy momentum tensor is derivable from a bare Lagrangian using canonical methods also. Appendix D contains a mathematical result to be used in the proof of Uniqueness Theorem.

[4.2]PRELIMINARIES

We shall work with the Lagrangian density of Eq.(4.1) and

shall use dimensional regularization throughout [6] We will use the minimal subtraction scheme [7] for renormalization of the theory as well as of composite operators [8].

The unrenormalized but dimensionally regularized Green's functions, connected Green's functions and proper vertices are generated respectively by W[J], Z[J] and $\Gamma[\Phi]$ with

$$W[J] = \frac{1}{N}\int D\phi \, \exp i\int d^n x \, [\, \mathcal{L} + J\phi] \tag{4.8}$$

where W[0] = 1.

$$Z[J] = -i \ln W[J]$$

$$\Phi(x) = \frac{\delta\Gamma}{\delta J(x)}$$

$$\Gamma[\Phi] = Z[J] - \int d^n x J(x)\Phi(x) \tag{4.9}$$

The renormalization transformations are defined by

$$\phi = Z^{1/2}\phi^R$$

$$m_o^2 = Z_m \, m^2$$

$$\lambda_o = \mu^{\varepsilon}\lambda Z_\lambda \tag{4.10}$$

In the M.S. scheme Z, Z_m and Z_λ are independent of m and μ and are functions only of λ and ε. The renormalization constants have the minimal subtraction form [7,9],

$$Z = 1+\sum_{r=1}^{\infty}\frac{Z^{(r)}(\lambda)}{\varepsilon^r} \tag{4.11}$$

where $\varepsilon = 4-n$.

The renormalized Green's functions, connected Green's functions and proper vertices are generated by $W^R[J^R]$, $Z^R[J^R]$ and $\Gamma^R[\Phi^R]$ respectively with

$$W^R[J^R] = W[J] \text{ and } J^R = JZ^{1/2} \tag{4.12}$$

$m_o^2\phi^2$ is a finite operator, since

$$\langle m_o^2\phi^2\rangle^{UR} = m_o^2\frac{\partial W}{\partial m_o^2} = m_o^2\frac{\partial m^2}{\partial m_o^2}\;\frac{\partial W^R}{\partial m^2}$$
$$= m^2\frac{\partial W}{\partial m^2}$$
$$= \langle m^2\phi^2\rangle^R \tag{4.13}$$

which implies that

$$\left\{\phi^2\right\}^R = Z_m\left\{\phi^2\right\}^{UR} \tag{4.14}$$

Consequently,

$$\left\{\partial^2\phi^2\right\}^R = Z_m\left\{\partial^2\phi^2\right\}^{UR} \tag{4.15}$$

We have the usual definitions of renormalization group[7],

$$\mu\,\frac{\partial\lambda}{\partial\mu}\bigg|_{\lambda_o,m_o,\varepsilon} \equiv \beta(\lambda,\varepsilon) = -\lambda\varepsilon + \beta_2\lambda^2 + O(\lambda^3)$$
$$\frac{1}{2}\,\mu\frac{\partial}{\partial\mu}\ln Z\bigg|_{\lambda_o,m_o,\varepsilon} \equiv \gamma(\lambda,\varepsilon) = \gamma(\lambda)$$
$$\frac{1}{2}\,\mu\frac{\partial}{\partial\mu}\ln Z_m\bigg|_{\lambda_o,m_o,\varepsilon} \equiv \gamma_m(\lambda,\varepsilon) = \gamma_m(\lambda) = \gamma_{m(1)}\lambda + O(\lambda^2) + \cdots \tag{4.16}$$

Above definition of $\beta(\lambda,\varepsilon)$, alongwith Eq.(4.10) implies that

$$\mu\frac{\partial}{\partial\mu}\,\ln Z_\lambda = -\frac{\beta(\lambda)}{\lambda} \tag{4.17}$$

[4.3]THE MOST GENERAL $\theta_{\mu\nu}$

In this section, we shall obtain the most general form of classically conserved energy momentum tensor for the $\lambda\phi^4$-theory,which is derivable from an action by cannonical procedure. We will show that it is indeed of the form in Eq.(4.5).

The most general energy momentum tensor is required to

have followong properties :

(1) $\partial^{\mu}\theta_{\mu\nu}$ vanishes if classical equations of motion are used. This implies that $\partial^{\mu}\theta_{\mu\nu}$ which is a Lorentz vector of dimension (n+1) and is local will have the form

$$\partial^{\mu}\theta_{\mu\nu} = a \frac{\partial S}{\partial\phi}\partial_{\nu}\phi + b\, \partial_{\nu}\left[\phi \frac{\partial S}{\partial\phi}\right] \tag{4.18}$$

(2) $\int\theta_{o\nu}d^3x$ gives the four momentum of the system correctly.

We will also assume it to be satisfying the following two conditions:

(i) It has terms with dimension n or less.

(ii) It is formally a finite function of bare quantities.

The reason for condition (ii) has already been discussed. Condition (i) has been imposed for simplicity's sake.

The most general $\theta_{\mu\nu}$ of dimension n is

$$\theta_{\mu\nu} = \alpha\partial_{\mu}\phi\partial_{\nu}\phi + \beta\phi\partial_{\mu}\partial_{\nu}\phi + g_{\mu\nu}(\gamma\partial_{\tau}\phi\, \partial^{\tau}\phi + \delta m_o^2\phi^2 + \eta\phi\partial^2\phi + \xi \frac{\lambda_o}{4!}\phi^4) \tag{4.19}$$

where α, β, γ, δ, η and ξ are constants.

Then,

$$\begin{aligned}\partial^{\mu}\theta_{\mu\nu} = & -(\alpha-\beta)\, \partial_{\nu}\phi \frac{\partial S}{\partial\phi} + \partial_{\nu}[(\gamma+\frac{\alpha+\beta}{2})(\partial_{\tau}\phi)(\partial^{\tau}\phi) + \delta m_o^2\phi^2 \\ & +(\eta+\beta)(\phi\partial^2\phi) + \xi \frac{\lambda_o}{4!}\phi^4 - (\frac{\alpha-\beta}{2})m_o^2\phi^2 - (\frac{\alpha-\beta}{4\,!})\lambda_o\phi^4]\end{aligned} \tag{4.20}$$

Requiring $\partial^{\mu}\theta_{\mu\nu}$ to have the form given in Eq (4.18), one obtains the following set of equations

$$\begin{aligned}&\gamma + \frac{\alpha+\beta}{2} = 0 \\ &\delta - \frac{\alpha-\beta}{2} = \eta + \beta = \frac{1}{4}(\xi - \alpha + \beta)\end{aligned} \tag{4.21}$$

Substituting for γ, δ and ξ in Eq (4.19) one obtains,

$$\theta_{\mu\nu} = A[\partial_\mu\phi\ \partial_\nu\phi - g_{\mu\nu}\mathcal{L}] + B[\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2]\phi^2 + Cg_{\mu\nu}\frac{\delta S}{\delta\phi}\phi \tag{4.22}$$

where A, B and C are constants given by

$$A = \alpha-\beta$$

$$B = \frac{\beta}{2}$$

$$C = -(\eta+\beta) \tag{4.23}$$

The first term in Eq.(4.22) is the canonical energy momentum tensor and the second term has the form of improvement term introduced by Callan, Coleman and Jackiw.

To determine the unknown coefficients, one may use the fact that $\int\theta_{o\nu}d^3x$ should give the four momentum of the system correctly. [It is not possilble to determine B in this manner as $B(\partial_o\partial_\nu - g_{o\nu}\partial^2)\phi^2$ does not contibute to $\int\theta_{o\nu}d^3x$ for any ν]. Thus,

$$P_\nu = \int \theta_{o\nu}d^3x = AP_\nu + Cg_{o\nu}\int\frac{\partial S}{\partial\phi}\phi d^3x$$

For $\nu = \iota$, the second term does not contribute and hence,

$$A = 1$$

This combined with the fact that the Green's functions of $\int g_{o\nu}\frac{\partial S}{\partial\phi}\phi d^3x$ are non-zero for ν=0 and q_o=0 implies that

$$C = 0$$

Hence, the energy momentum tensor satisfying the properties (1) and (2) and conditions (i) and (ii) is

$$\theta_{\mu\nu} = \theta^c_{\mu\nu} + B\ (\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2)\phi^2$$

where B is a finite function of bare quantities. Now B is dimensionless and by assumption can be a function of $\lambda_o\mu^{-\varepsilon}$, m_o^2/μ^2 and ε only. Thus,we obtain the form of Eq.(4.5) where $H(\lambda_o\mu^{-\varepsilon}, m_o^2/\mu^2, \varepsilon)$ = B is a finite function of its arguments.

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[4 4]PROOF OF UNIQUENESS

We shall now show that the energy momentum tensor of Eq.(4.5), if it has finite matrix elements to all orders in perturbation theory, is unique. Eq. (4.5) can be rewritten as

$$\theta_{\mu\nu} = \theta^{C}_{\mu\nu} + \frac{F'(\lambda_o\mu^{-\varepsilon}, m_o^2/\mu^2, \varepsilon)}{1-n}(\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2)\phi^2 \qquad (4.24)$$

where $F'(\lambda_o\mu^{-\varepsilon}, m_o^2/\mu^2, \varepsilon) = (1-n)H(\lambda_o\mu^{-\varepsilon}, m_o^2/\mu^2, \varepsilon)$ is also a finite function of its arguments [We have reparametrized $H(\lambda_o\mu^{-\varepsilon}, m_o^2/\mu^2, \varepsilon)$ of Eq(4.5) for future convenience].

We have already shown in Sec.(3.2) that $\theta_{\mu\nu}$ is finite if and only if $\theta_\mu{}^\mu$ is a finite operator. Therefore, it is sufficient to consider the finiteness of $\theta_\mu{}^\mu$. $\theta_\mu{}^\mu$ is given by

$$\theta_\mu{}^\mu = \theta^{C\mu}_\mu + F'(\lambda_o\mu^{-\varepsilon}, m_o^2/\mu^2, \varepsilon)\partial^2\phi^2 \qquad (4.25)$$

As shown by Collins [2], there is at least one energy momentum tensor $\bar{\theta}_{\mu\nu}$ of Eq.(4.7) which has finite matrix elements to all orders. It is convenient to take the difference

$$\begin{aligned} \langle\theta_\mu{}^\mu\rangle - \langle\bar{\theta}_\mu{}^\mu\rangle &= [F' - (1-n)H_o(\varepsilon)]\langle\partial^2\phi^2\rangle \\ &\equiv F(\lambda_o\mu^{-\varepsilon}, m_o^2/\mu^2, \varepsilon)\langle\partial^2\phi^2\rangle \\ &\equiv F(\lambda_o\mu^{-\varepsilon}, m_o^2/\mu^2, \varepsilon)Z_m^{-1}\{\partial^2\phi^2\}^R \end{aligned} \qquad (4.26)$$

Proving the uniqueness amounts to showing that if the right hand side of Eq.(4.26) is finite, then it must vanish. As $\{\partial^2\phi^2\}^R$ is a finite operator, this amounts to showing that if

$$F(\lambda_o\mu^{-\varepsilon}, m_o^2/\mu^2, \varepsilon)Z_m^{-1} = \text{finite} = F(\lambda Z_\lambda, \frac{m^2 Z_m}{\mu^2}, \varepsilon)Z_m^{-1} \qquad (4.27)$$

then,

$$F(\lambda_o\mu^{-\varepsilon}, m_o^2/\mu^2, \varepsilon) = 0 \qquad (4.28)$$

The proof of the above is made much more complicated as compared to the proof of the uniqueness theorem of Collins [2] by the presence of Z_λ and Z_m in the arguments of F in Eq.(4.27).

We expand F in Taylors series in $\frac{m^2}{\mu^2} Z_m$[1] :

$$F'(\lambda_0 \mu^{-\varepsilon}, m_0^2/\mu^2, \varepsilon) = \sum_{q=0}^{\infty} f_q(\lambda Z_\lambda, \varepsilon)(m^2 Z_m/\mu^2)^q \tag{4.29}$$

Eq.(4.27) then becomes

$$\sum_{q=0}^{\infty} f_q(Z_\lambda \lambda\ ,\varepsilon)(\frac{m^2}{\mu^2})^q\ Z_m^{q-1}(\lambda,\varepsilon) = \text{finite} \tag{4.30}$$

This requires that for each q,

$$f_q(\lambda Z_\lambda, \varepsilon) Z_m^{q-1} = \text{finite}, \qquad q=0,1,2,\cdots\cdots \tag{4.31}$$

Thus the uniqueness will be proved if we prove the following theorem:

<u>Theorem</u> : The only solution to

$$f(\lambda Z_\lambda, \varepsilon) Z_m^{q-1} = \text{finite} \tag{4.32}$$

where f is a power series in non-negative powers of λZ_λ and ε, is that

---

[1]We can assume without loss of generality that F' of Eq.(4.26) is a polynomial in m^2/μ^2. This is so because we want $\theta^{c\mu}$ to be finite. Thus the divergence coming from $\theta^{c\mu}$ must be cancelled by that coming from $F'\partial^2\phi^2$. But in minimal subtraction scheme of dimensional regularization, the divergences coming from $\theta^{c\mu}$ and $\partial^2\phi^2$ are both polynomials in m^2. A further restriction along these lines could be used to restrict F', but this turns out to be unnecessary.

$$f \equiv 0 \tag{4.33}$$

for q any non-negative integer.

Proof : The proof proceeds for any fixed q as follow:.

Eq.(4.32) can be rewritten by expanding f

$$f Z_m^{q-1} \equiv \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} f_{rs} \lambda^r \varepsilon^s Z_\lambda^r \; Z_m^{q-1} = \text{finite} \tag{4.34}$$

$Z_\lambda^r \; Z_m^{q-1}$ has the " minimal-subtraction form "

$$Z_\lambda^r Z_m^{q-1} \equiv Y_r \equiv 1 + \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} Y_{r,a}^b \lambda^a \varepsilon^{-b} \tag{4.35}$$

Comparing the coefficient of $\dfrac{\lambda^l}{\varepsilon^{l-p}}$, for $l > p$, on both sides of Eq(4.34) one obtains

$$\sum_{r} \sum_{\substack{s \\ r+s \le p}} f_{rs} Y_{r,l-r}^{l+s-p} = 0, \qquad p=0,\ 1,\ 2, \ldots\ l-1 \tag{4.36}$$

The constraint $r+s \le p$ follows from the fact that

$$Y_{r,a}^b = 0 \quad \text{if} \quad b > a.$$

We consider the equation for $p=0$, $l=1$. We have

$$f_{00} Y_{0,1}^1 = 0 \tag{4.37}$$

It is easy to verify that $Y_{0,1}^1 \neq 0$. Hence

$$f_{00} = 0 \tag{4.38}$$

We now proceed by induction on p. Let $f_{rs} = 0$ for $r+s = 0, 1, \ldots, p-1$. Then, we shall show from Eq.(4.36) that $f_{rs} = 0$ for $r+s = p$. We have already verified the assumption for $p = 0$ on account of Eq.(4.38).

Using $f_{rs} = 0$ for $r+s < p$ in Eq. (4.36), we have

$$\sum_{r=0}^{p} f_{r,p-r} Y_{r,l-r}^{l-r} \equiv \sum_{r=0}^{p} f_{r,p-r} C_{l-r}^r = 0 \tag{4.39}$$

Here $C_{l-r}^r = Y_{r,l-r}^{l-r}$ = the coefficient of $\dfrac{\lambda^{l-r}}{\varepsilon^{l-r}}$ in $Z_\lambda^r Z_m^{q-1}$.

Eq.(4.39) is valid for an infinite set of values of l starting

from $\iota = p+1$. (Recall: $\iota > p$). Out of this infinite set of equations, it proves sufficient to consider first (p+1) equations i.e. with $\iota = p+1, p+2, \ldots, 2p+1$. We wish to show that $f_{r,p-r} = 0;\ 0 \le r \le p$ is the only solution allowed for these equations. This will be so if we show that the determinant Δ of coefficients,

$$\Delta = \begin{vmatrix} C^0_{p+1} & C^1_p & C^2_{p-1} & \cdots\cdots & C^p_1 \\ C^0_{p+2} & C^1_{p+1} & C^2_p & \cdots\cdots & C^p_2 \\ \vdots & \vdots & \vdots & & \vdots \\ C^0_{2p+1} & C^1_{2p} & C^2_{2p-1} & & C^p_{p+1} \end{vmatrix} \tag{4.40}$$

is non-zero. This is what we set out to prove. Δ can be simplified further by noting that relations exist between elements of each column. To see this, consider Eqs.(4.16) and (4.17). These lead to

$$\mu\frac{\partial}{\partial\mu}(Y_r) = \mu\frac{\partial}{\partial\mu}(Z^r_\lambda\ Z^{q-1}_m) = -[2\gamma_m(q-1)+r\frac{\beta(\lambda)}{\lambda}]Y_r \tag{4.41}$$

Using Eq.(4.16), this becomes

$$[-\lambda\varepsilon + \beta(\lambda)]\frac{\partial}{\partial\lambda}Y_r = -[2\gamma_m(q-1) + r\frac{\beta(\lambda)}{\lambda}]Y_r \tag{4.42}$$

Substituting the expansion of Eq.(4.35) in Eq.(4.42) and comparing the coefficients of $\dfrac{\lambda^s}{\varepsilon^{s-1}}$ one obtains

$$C^r_s = \frac{1}{s}\ [(r+s-1)\beta_2 + 2\gamma_{m(1)}(q-1)]C^r_{s-1} \tag{4.43}$$

Now using[10],

$$\beta_2 = \frac{3}{16\pi^2} \qquad \text{and} \qquad 2\gamma_{m(1)} = \frac{1}{16\pi^2} \tag{4.44}$$

one verfies that the square bracket in Eq.(4.42) does not

vanish for any non-negative integer q and any positive integers r and s. The remarkable thing about relation (4.43) is that the square bracket on the right hand side depends only on (r+s). This has the immediate consequence that Δ can be written as

$$\Delta = C_1 \Delta^{(0)} \tag{4.45}$$

where

$$C_1 = [p\beta_z + 2\gamma_{m(1)}(q-1)][(p+1)\beta_z + 2\gamma_{m(1)}(q-1)]\cdots\cdots [2p\beta_z + 2\gamma_{m(1)}(q-1)]C^0_p\ C^1_{p-1}\ldots C^p_0 \tag{4.46}$$

Using Eq.(4.43) again one obtains

$$C_1 = K\ C^0_1\ C^1_1\ C^2_1\ \ldots\ C^p_1$$

where K is a product of terms like $\frac{1}{s}[(r+s-1)\beta_z + (q-1)2\gamma_{m(1)}]$ and $[p\beta_z + 2\gamma_{m(1)}(q-1)]$, ... $[2p\beta_z + 2\gamma_{m(1)}(q-1)]$ all of which are non-zero. Hence

$$K \neq 0.$$

One can easily show by direct calculation that

$$C^r_1 = Y^1_{r,1} \neq 0 \text{ for each r.}$$

Thus $C_1 \neq 0$.

$\Delta^{(0)}$ is a (p+1) x (p+1) matrix that depends purely on p in a simple manner viz.

$$\Delta^{(0)} = \begin{vmatrix} 1 & 1 & 1 & \cdots\cdots & 1 \\ \frac{1}{p+2} & \frac{1}{p+1} & \frac{1}{p} & \cdots\cdots & \frac{1}{2} \\ \frac{1}{(p+2)(p+3)} & \frac{1}{(p+1)(p+2)} & \frac{1}{p(p+1)} & \cdots\cdots & \frac{1}{2.3} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{1}{(p+2)\ldots(2p+1)} & \frac{1}{(p+1)\ldots 2p} & \frac{1}{p\ldots(2p-1)} & \cdots\cdots & \frac{1}{(p+1)!} \end{vmatrix} \tag{4.47}$$

We show in Appendix D that $\Delta^{(0)} \neq 0$ for any positive integer p. Hence the system of equations (4.39) have the solutions that

$$f_{r,p-r} = 0, \qquad r = 0, 1, \ldots\ldots, p$$

This completes the proof by induction on p. Hence the theorem expressed by Eqs.(4.32) and (4.33) is proved. This, in turn, proves uniqueness of $\bar{\theta}_{\mu\nu}$ of Eq. (4.7).

[4.5]DERIVABILITY FROM A BARE ACTION

It is well known that the improved energy momentum tensor in Eq.(4.5) can be obtained from a modified bare action using

$$\theta_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}}\bigg|_{g^{\mu\nu}=\eta^{\mu\nu}}$$

We will now show that it can also be derived from a bare action using canonical methods. This action contains, apart from fields and their derivatives, second derivatives of fields also:

$$S = \int d^n x\, \mathscr{L}$$

where

$$\mathcal{L} = \mathcal{L}(\phi, \partial_\mu\phi, \partial_\mu\partial_\nu\phi) \tag{4.48}$$

Adding second derivative terms to the Lagrangian density changes the equations of motion. Requiring the action to be stationary for fields which are solutions of actual equations of motion, we obtain the energy momentum tensor $T_{\mu\alpha}$ defined by

$$T^\mu{}_\alpha = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\alpha\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi)}\,\partial_\alpha\partial_\nu\phi + \partial_\nu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi)}\right]\partial_\alpha\phi - g_\alpha{}^\mu\mathcal{L} \tag{4.49}$$

which is the most general energy momentum tensor derived from a Lagrangian containing second derivatives of fields. Now, substituting for $\mathcal{L}$ the following Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\,m_o^2\phi^2 - \frac{\lambda_o\phi^4}{4!} + H_o(\lambda_o\mu^{-\varepsilon}, \frac{m_o^2}{\mu^2}, \varepsilon)\partial^2\phi^2 \tag{4.50}$$

one straightforwardly obtains the energy momentum tensor in Eq.(4.5).

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CHAPTER 5

ENERGY MOMENTUM TENSOR IN SCALAR QUANTUM ELECTRODYNAMICS

[5.1]INTRODUCTION

It is well known that the energy momentum tensor in gauge theories without scalars has finite and gauge independent matrix elements. In scalar theory, the energy momentum tensor derived from minimal Einstein action is not finite but one can obtain an improved energy momentum tensor which is finite to all orders in perturbation theory[1]. This energy momentum tensor is the unique finite energy momentum tensor and it can be derived from an action which is a finite function of bare parameters of the theory[2].The procedure of obtaining finite energy momentum tensor by adding to it improvement terms of the special kind described in chapter 3 is called finite improvement program.

Since the finite improvement program is found to be successful in $\lambda\phi^4$-theory, one may naturally raise the question if a similar finite improvement program will work in other renormalizable theories involving scalars also.Now we will analyze this question in the context of some renormalizable theories involving scalar fields and having two coupling constants. These are

(i) Scalar quantum electrodynamics (Scalar Q.E.D.)

(ii)Non-abelian gauge theory with scalars (NAGT's with scalars)

(iii)Yukawa theory

(iv) A theory of two interacting scalar fields

Each of these theories contains apart from the self coupling of scalar field λ another coupling constant, say $\varkappa$. In the limit $\varkappa \to 0$, each of these theories effectively reduces to a decoupled scalar ϕ^4- theory. Obviously, the energy momentum tensor derived from the minimal Einstein action (i.e. one without any improvement term)does not have finite matrix elements in $O(\varkappa^0)$ itself and therefore an improvement term is necessarily needed. The question arises whether a finite improvement program similar to the one obtained by Collins for scalar ϕ^4-theory will work or not. It is possible that the improvement term obtained by Collins may suffice for these theories as well. However, as we have verified , using explicit calculations[3,4], this is not the case. Collins' finite improvement program does work at $\varkappa = 0$, but for higher orders in $\varkappa$ it simply breaks down in the sense that in sufficiently high orders (for example $O(\lambda^2 e^2)$ or $O(\lambda e^4)$ in scalar Q.E.D.) Collins' energy momentum tensor does not have finite matrix elements thus necessitating the need for some further improvement. One may recall that Collin's improvement coefficient depends on ε only. There is still a possilbility of a more general improvement program in which the improvement coefficient depends, in addition to ε, also on the two coupling constants of the theory . It is this possibility that we shall

explore in the following chapters.

As has already been discussed in detail in chapter 3, there can be two different interpretations of the finite improvement program[2,3]:

(i) One, in which the improvement coefficient is a finite function of bare quantities at $\varepsilon=0$.

(ii) Second, in which the improvement coefficient is a finite quantity i.e. a finite function of renormalized quantities at $\varepsilon=0$.

Both of these programs, if they succeed, will lead to a finite energy momentum tensor without introducing any new infinite counterterms apart from those of the flat space theory. However, in case neither of these programs work then, to obtain finite matrix elements for $\theta_{\mu\nu}$, one would necessarily have to add infinite counterterms and hence extra information from experiment would be needed. Here, we will show that it is, in fact, impossible (in all the four cases stated above) to construct a finite energy momentum tensor in either of these two ways. We shall start with scalar Q.E.D. in this chapter and then proceed on to other cases in next chapters.

[5.2]PRELIMINARIES

We shall work with a complex scalar field coupled to an abelian gauge field described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^* (D^\mu\phi) - m_o{}^2\phi^*\phi - \frac{\lambda_o}{4}(\phi^*\phi)^2 - \frac{1}{2}\xi_o(\partial.A)^2 \tag{5.1}$$

where $D_\mu\phi$ is the covariant derivative defined by

$$D_\mu\phi = \partial_\mu\phi - \iota\, e_o A_\mu\phi$$

and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\nu$$

We shall work with dimensionally regularized quantities and will use the minimal subtraction scheme [5,6]. The unrenormalized but dimensionally regularized Green's functions, connected Green's functions and proper vertices are generated respectively by $W[J,J^*,J_\mu]$, $Z[J,J^*,J_\mu]$ and $\Gamma[\Phi,\Phi^*,\mathcal{A}_\mu]$ with

$$W[J,J^*,J_\mu] = \frac{1}{N}\int \mathcal{D}A_\mu \mathcal{D}\phi\, \mathcal{D}\phi^* \exp \iota\int d^nx[\mathcal{L} + J^*\phi + J\phi^* + J_\mu A^\mu] \tag{5.2}$$

where $W[0] = 1$.

$$Z[J,J^*,J_\mu] = -\iota \ln W[J,J^*,J_\mu]$$

$$\Phi(x) = \frac{\delta Z}{\delta J^*(x)}$$

$$\mathcal{A}_\mu(x) = \frac{\delta Z}{\delta J^\mu(x)}$$

$$\Gamma[\Phi,\Phi^*,\mathcal{A}_\mu] = Z[J,J^*,J_\mu] - \int d^nx[J^*(x)\Phi(x) + J(x)\Phi^*(x) + J_\mu(x)\mathcal{A}^\mu(x)] \tag{5.3}$$

In the M.S. scheme the renormalization parameters are

$$\phi = Z^{1/2}\phi^R \qquad m_o^2 = m^2 Z_m$$

$$\lambda_o = \mu_1^\varepsilon\,[\lambda Z_\lambda + \delta\lambda] \qquad e_o^2 = \mu_2^\varepsilon\, e^2\, Z_e^2$$

$$A_\mu = Z_3^{1/2}\, A_\mu^R$$

$$\xi_o = Z_\xi\, \xi = Z_3^{-1}\xi \tag{5.4}$$

where μ_1 and μ_2 are two <u>independent</u> arbitrary dimensional parameters. [Note that there are two mass scales μ_1 and μ_2 due to the presence of two coupling constants. One can choose μ_1 =

where $D_\mu\phi$ is the covariant derivative defined by

$$D_\mu\phi = \partial_\mu\phi - \iota\, e_o A_\mu \phi$$

and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

We shall work with dimensionally regularized quantities and will use the minimal subtraction scheme [5,6]. The unrenormalized but dimensionally regularized Green's functions, connected Green's functions and proper vertices are generated respectively by $W[J,J^*,J_\mu]$, $Z[J,J^*,J_\mu]$ and $\Gamma[\Phi,\Phi^*,\mathcal{A}_\mu]$ with

$$W[J,J^*,J_\mu] = \frac{1}{N}\int \mathcal{D}A_\mu \mathcal{D}\phi\, \mathcal{D}\phi^* \exp\iota\int d^nx[\mathcal{L} + J^*\phi + J\phi^* + J_\mu A^\mu] \tag{5.2}$$

where $W[0] = 1$.

$$Z[J,J^*,J_\mu] = -\iota \ln W[J,J^*,J_\mu]$$

$$\Phi(x) = \frac{\delta Z}{\delta J^*(x)}$$

$$\mathcal{A}_\mu(x) = \frac{\delta Z}{\delta J^\mu(x)}$$

$$\Gamma[\Phi,\Phi^*,\mathcal{A}_\mu] = Z[J,J^*,J_\mu] - \int d^nx[J^*(x)\Phi(x) + J(x)\Phi^*(x) + J_\mu(x)\mathcal{A}^\mu(x)] \tag{5.3}$$

In the M.S. scheme the renormalization parameters are

$$\phi = Z^{1/2}\phi^R \qquad m_o^2 = m^2 Z_m$$

$$\lambda_o = \mu_1^\varepsilon\,[\lambda Z_\lambda + \delta\lambda] \qquad e_o^2 = \mu_2^\varepsilon\, e^2\, Z_e^2$$

$$A_\mu = Z_3^{1/2} A_\mu^R$$

$$\xi_o = Z_\xi\, \xi = Z_3^{-1}\xi \tag{5.4}$$

where μ_1 and μ_2 are two <u>independent</u> arbitrary dimensional parameters. [Note that there are two mass scales μ_1 and μ_2 due to the presence of two coupling constants. One can choose μ_1 =

$\mu_2 = \mu$, but we have chosen μ_1 and μ_2 different for the sake of generality].

One may notice that the renormalization of λ is different from $\lambda\phi^4$- theory where λ is multiplicatively renormalizable. The reason lies in the coupling of scalar field with gauge field. ϕ^4 vertex receives contributions from diagrams which are $O(e^{2n})$ $n \geq 2$. Such contributions are not proportional to λ and hence one has to add $\delta\lambda$ in the renormalization transformation. $\delta\lambda$ starts with $O(e^4)$.

The renormalization constants have the minimal subtraction form [5,6,7]

$$Z = 1 + \sum_{r=1}^{\infty} \frac{Z^{(r)}(\lambda, e^2, c)}{\varepsilon^r} \tag{5.5}$$

where $\varepsilon = 4-n$ and $c = \mu_1/\mu_2$ [Due to the presence of two mass scales, the renormalization constants in mass independent scheme will depend on powers of ln c also].

The renormalized Green's functions, connected Green's functions and proper vertices are generated by $W^R[J^R, J^{*R}, J^R_\mu]$, $Z^R[J^R, J^{*R}, J^R_\mu]$ and $\Gamma^R[\Phi^R, \Phi^{*R}, \mathscr{A}_\mu]$ respectively with

$$W^R[J^R, J^{*R}, J^R_\mu] = W[J, J^*, J_\mu] \text{ and } J^R = Z^{1/2} J \text{ etc.}$$

Equations of motion imply that

$$\begin{aligned}
\left\langle \frac{\delta S}{\delta\phi}\phi + \phi^*\frac{\delta S}{\delta\phi^*} \right\rangle &= - \langle J^*\phi + J\phi^* \rangle \\
&= - J^{*R}\langle\phi\rangle^R - J\langle\phi^*\rangle^R \\
&= \left\langle \frac{\delta S}{\delta\phi}\phi + \phi^*\frac{\delta S}{\delta\phi^*} \right\rangle^R = \text{finite}
\end{aligned} \tag{5.6}$$

Similarly

$$\left\langle \frac{\delta S}{\delta\mathscr{A}_\mu}\mathscr{A}_\mu \right\rangle^R = \left\langle \frac{\delta S}{\delta\mathscr{A}_\mu}\mathscr{A}_\mu \right\rangle = \text{finite} \tag{5.7}$$

Also, as shown in Sec. [4.4] for $\lambda\phi^4$-theory, here also

$$\langle m_o^2\phi^*\phi\rangle^R = \langle m_o^2\phi^*\phi\rangle^{U.R} = \text{finite} \tag{5.8}$$

Thus $\frac{\delta S}{\delta\phi}\phi + \phi^*\frac{\delta S}{\delta\phi^*}$, $\frac{\delta S}{\delta A_\mu}A_\mu$ and $m_o^2\phi^*\phi$ are finite operators. $\partial^2(\phi^*\phi)$ is a multiplicatively renormalizable operator just as $\partial^2\phi^2$ is mutiplicatively renormalizable operator in $\lambda\phi^4$- theory [see Eq.(4.15)],

$$\left\{\partial^2(\phi^*\phi)\right\}^{U.R} = Z_m^{-1}\left\{\partial^2(\phi^*\phi)\right\}^R \tag{5.9}$$

We shall use renormalization group extensively. Due to the presence of two mass scales and two coupling constants there is a larger number of renormalization group functions here as compared to $\lambda\phi^4$-theory. For example, one may consider variation of λ with respect to both μ_1 and μ_2 thus obtaining two β functions - β_1^λ and β_2^λ. Similarly, there will be two β functions β_1^e and β_2^e corresponding to e also. If we had chosen $\mu_1 = \mu_2 = \mu$, there would have been only two β functions β^λ and β^e, but that would not have changed our analysis because

$$\beta^\lambda = (\beta_1^\lambda + \beta_2^\lambda)\Big|_{\mu_1=\mu_2}$$

and $$\beta^e = (\beta_1^e + \beta_2^e)\Big|_{\mu_1=\mu_2}$$

Similarly, one can define other renormalization group functions also. Below we give the definitions of various renormalization group functions and their dependence on ε (or its lack)[1]. (We have also indicated the leading terms which are needed in future discussion) :

[1] In principle β^e could have a term of $_2O(e\lambda)$. But $Z = Z^{-1/2}$ because of WT identity; Now, $Z = 1 + O(e^2)$, because at e = 0 there is no renormalization of photon propogator as it is a free field at e=0.

We define

$$\beta_1^{\lambda}(\lambda,e,c,\varepsilon) \equiv \mu_1\frac{\partial\lambda}{\partial\mu_1}\bigg|_{\lambda_0,e_0,m_0^2,\xi_0,\varepsilon,\mu_2} = -\lambda\varepsilon + \beta_1^{\lambda}(\lambda,e,c)$$

$$= -\lambda\varepsilon + \beta_2\lambda^2 + \beta_3\lambda^3 + \cdots\cdots$$

$$+ \beta_2' e^4 + \cdots\cdots$$

$$\beta_2^{\lambda}(\lambda,e,c,\varepsilon) \equiv \mu_2\frac{\partial\lambda}{\partial\mu_2}\bigg|_{\lambda_0,e_0,m_0^2,\xi_0,\varepsilon,\mu_1} = \beta_2^{\lambda}(\lambda,e,c)$$

$$\gamma_{m_1}(\lambda,e,c,\varepsilon) \equiv \frac{1}{2}\mu_1\frac{\partial}{\partial\mu_1}\ln Z_m\bigg|_{\lambda_0,e_0,m_0^2,\xi_0,\varepsilon,\mu_2} = \gamma_{m_1}(\lambda,e,c)$$

$$\equiv \frac{1}{2}\left[g_0'(\lambda) + e^2 g_2'(\lambda) + \cdots\cdots\cdots\right.$$

$$= \frac{1}{2}\left[\gamma_{m_1}^{(1)}\lambda + \cdots\cdots\right] + O(e^2) + \cdots\cdots$$

$$\gamma_{m_2}(\lambda,e,c,\varepsilon) \equiv \frac{1}{2}\mu_2\frac{\partial}{\partial\mu_2}\ln Z_m\bigg|_{\lambda_0,e_0,m_0^2,\xi_0,\varepsilon,\mu_1} = \gamma_{m_2}(\lambda,e,c)$$

$$= e^2\gamma_{m_2}^{(0)} + \cdots\cdots$$

$$\gamma_1(\lambda,e,c,\varepsilon,\xi) \equiv \frac{1}{2}\mu_1\frac{\partial}{\partial\mu_1}\ln Z\bigg|_{\lambda_0,e_0,m_0^2,\varepsilon,\mu_2} = \gamma_1(\lambda,e,c,\xi)$$

$$\gamma_2(\lambda,e,c,\varepsilon,\xi) \equiv \frac{1}{2}\mu_2\frac{\partial}{\partial\mu_2}\ln Z\bigg|_{\lambda_0,e_0,m_0^2,\varepsilon,\mu_1} = \gamma_2(\lambda,e,c,\xi)$$

$$\gamma_1'(\lambda,e,c,\varepsilon) \equiv \frac{1}{2}\mu_1\frac{\partial}{\partial\mu_1}\ln Z_3\bigg|_{\lambda_0,e_0,m_0^2,\xi_0,\varepsilon,\mu_2} = \gamma_1'(\lambda,e,c)$$

$$\gamma_2'(\lambda,e,c,\varepsilon) \equiv \frac{1}{2}\mu_2\frac{\partial}{\partial\mu_2}\ln Z_3\bigg|_{\lambda_0,e_0,m_0^2,\xi_0,\varepsilon,\mu_1} = \gamma_2'(\lambda,e,c)$$

$$\gamma_{\xi}^{1}(\lambda,e,c,\varepsilon) \equiv \frac{1}{2}\mu_1\frac{\partial}{\partial\mu_1}\ln\xi\bigg|_{\lambda_0,e_0,m_0^2,\xi_0,\varepsilon,\mu_2} = \gamma_{\xi}^{1}(\lambda,e,c)$$

$$\gamma_{\xi}^{2}(\lambda,e,c,\varepsilon) \equiv \frac{1}{2}\mu_2\frac{\partial}{\partial\mu_2}\ln\xi\bigg|_{\lambda_0,e_0,m_0^2,\xi_0,\varepsilon,\mu_1} = \gamma_{\xi}^{2}(\lambda,e,c)$$

$$\beta_1^{e}(\lambda,e,c,\varepsilon) \equiv \frac{1}{2}\mu_1\frac{\partial e}{\partial\mu_1}\bigg|_{\lambda_0,e_0,m_0^2,\xi_0,\varepsilon,\mu_2} = \beta_1^{e}(\lambda,e,c)$$

$$= e^3\left[h^{(2)}(\lambda) + e^2 h^{(4)}(\lambda) + \cdots\cdots\cdots\right]$$

where $h^{(2)}(\lambda) = h_1^{(2)}(\lambda)\lambda + \cdots\cdots\cdots$

$$\beta_2^e(\lambda, e, c, \varepsilon) \equiv \frac{1}{2}\,\mu_2 \frac{\partial e}{\partial \mu_2}\bigg|_{\lambda_0, e_0, m_0^2, \xi_0, \varepsilon, \mu_1} = -\frac{e\varepsilon}{2} + \beta_2^e(\lambda, e, c)$$

$$= -\frac{e\varepsilon}{2} + O(e^3) \qquad (5.10)$$

We shall need the renormalization of the operator $O_1 = -\frac{\lambda(\phi^*\phi)^2}{4} + \frac{1}{4}\, FF + \frac{1}{2}\,\xi_0 (\partial . A)^2$.
The complete set of operators with which it can mix under renormalization are given below[2]:

$$O_1 = -\frac{\lambda_0}{4}(\phi^*\phi)^2 + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2}\,\xi_0 (\partial . A)^2$$

$$O_3 = \phi^* \frac{\delta S}{\delta \phi^*} + \frac{\delta S}{\delta \phi}\,\phi$$

$$O_4 = \frac{\delta S}{\delta A_\mu(x)}\, A_\mu(x)$$

$$O_5 = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2}\,\xi_0 (\partial . A)^2$$

$$O_6 = \frac{1}{2}\,\xi_0 (\partial . A)^2$$

$$O_7 = \partial^2 (\phi^*\phi) \qquad (5.11)$$

Given a complete set of operators that mix with each other under renormalization, one defines the renormalization matrix

$$\langle O_i \rangle^{U.R.} = \Sigma\, Z_{ij} \langle O_j \rangle^{R} \qquad (5.12)$$

where $\langle O_i \rangle^{R}$ and $\langle O_i \rangle^{U.R.}$ are respectively the renormalized and

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[2] In the set of operators in Eq.(5.11) only the combination $\frac{\delta S}{\delta \phi}\phi + \frac{\delta S}{\delta \phi^*}\phi^*$ appears and not $\frac{\delta S}{\delta \phi}\phi$ and $\frac{\delta S}{\delta \phi^*}\phi^*$ individually. This is because all other operators and the action are symmetric under $\phi \longleftrightarrow \phi^*$ and $A \longrightarrow -A$. Hence, only this combination which is symmetric under this operation appears in the renormalization counterterms of the operators in Eq.(5.11).

unrenormalized Green's functions of the operator O_i. The renormalized operators in the M.S. scheme are defined straightforwardly for operators O_2 - O_6 at zero momentum :

$$\langle \int d^n x\ O_2^R \rangle = -m^2 \frac{\partial Z^R}{\partial m^2}$$

$$\langle \int d^n x\ O_3^R \rangle = -\int d^n x [J^R \frac{\partial Z^R}{\partial J^R} + J^{*R} \frac{\delta Z^R}{\delta J^{*R}}]$$

$$\langle \int d^n x\ O_4^R \rangle = -\int d^n x\ J_\mu^R \frac{\delta Z^R}{\delta J_\mu^R}$$

$$\langle \int d^n x\ O_5^R \rangle = \frac{1}{2} e \frac{\partial Z^R}{\partial e} + \frac{1}{2} \int d^n x J_\mu{}^R \frac{\delta Z^R}{\delta J_\mu^R}$$

$$\langle \int d^n x\ O_6^R \rangle = - \xi \frac{\partial Z^R}{\partial \xi} \qquad (5.12)$$

Renormalization of O_7 has already been defined in Eq.(5.9).However, there is a subtlety involved in renormalizing O_1. One may use

$$\int d^n x\ O_1 = \lambda_o \frac{\partial S}{\partial \lambda_o} + \frac{1}{2} e_o \frac{\partial S}{\partial e_o} - \frac{1}{2} \int d^n x\ A_\mu(x) \frac{\delta S}{\delta A_\mu(x)}$$

to define

$$\langle \int d^n x\ O_1^R \rangle = \lambda \frac{\partial Z^R}{\partial \lambda} + \frac{1}{2} e \frac{\partial Z^R}{\partial e} + \frac{1}{2} \int d^n x\ J_\mu{}^R \frac{\delta Z^R}{\delta J_\mu{}^R} \qquad (5.12a)$$

But,this definition is valid only upto $O(e^2\lambda^n)$ for arbitrary n.In $O(e^4)$ a problem arises because λ_o does not vanish at $\lambda = 0$ Due to the presence of $\delta\lambda$ in the third of Eqs.(5.4), it is no longer true that

$$\langle \lambda_o \frac{\partial S}{\partial \lambda_o} \rangle^R = \lambda \frac{\partial Z^R}{\partial \lambda}$$

as the right hand side vanishes at $\lambda = 0$, whereas the left hand side does not vanish from $O(e^4)$ onwards. But we can avoid the question of defining $\langle O_1 \rangle^R$ by defining a renormalization

matrix by

$$\langle O_i \rangle^{U.R.} = \sum_j Z_{ij} \langle X_j \rangle^R \tag{5.13}$$

where X_1 is defined by the right hand side of Eq.(5.12a) and $X^R_j = O^R_j$ for $j = 2,3,\cdots,7$. As explained in Appendix G, Z_{ij} continues to be a polynomial in the loop expansion parameter a and is invertible. From Eq.(5.13) and the following properties

(a) $\langle O_j \rangle^{U.R}$ are independent of μ_1 and μ_2

(b) $\langle X_j \rangle^R$ are finite

(c) $\langle X^R_j \rangle$ are linearly independent functionals

(d) Z_{ij} is invertible matrix,

it still follows that

$$Z^{-1}_{ij}\ \mu_1 \frac{\partial}{\partial \mu_1} Z_{jk} = \gamma_{ik} = \text{finite at } \varepsilon = 0 \tag{5.14}$$

(Since this is all that we shall use, the question of defining O^R_1 can be evaded. Moreover, our results need only $O(e^2)$ quantities and to this order the difference between O^R_1 and X^R_1 turns out to have no consequence). One can deduce the structure of $\mathbb{Z}$ using Eqs.(5.6)-(5.9) and the result in Appendix F :

$$\mathbb{Z} = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} & Z_{15} & Z_{16} & Z_{17} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ Z_{51} & Z_{52} & Z_{53} & Z_{54} & Z_{55} & Z_{56} & Z_{57} \\ 0 & 0 & Z_{63} & Z_{64} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & Z^{-1}_m \end{bmatrix} \tag{5.15}$$

where Z_{1j} $(j = 1,2,\cdots,7)$ are as yet unknown. We shall determine

Z_{1j} (j = 1, 2, ..., 6) in Sec.[5.4] using the technique of Ref.8.

The mass renormalization constant Z_m^{-1} can be expanded in powers of e^2:

$$Z_m^{-1} = Z_{m(0)}^{-1} + e^2 Z_{m(2)}^{-1} + e^4 Z_{m(4)}^{-1} + \cdots \qquad (5.16)$$

We define A_{rs} by

$$Z_{m(0)}^{-1} = 1 + \sum_{r=1}^{\infty}\sum_{s=1}^{r} \frac{A_{rs}}{\varepsilon^r} \lambda^r \qquad (5.17)$$

Z_m^{-1} saitsfies the following equations :

$$\mu_1 \frac{\partial}{\partial \mu_1} Z_m^{-1} = \mu_1 \frac{\partial \lambda}{\partial \mu_1}\bigg|_{bare} \frac{\partial Z_m^{-1}}{\partial \lambda} + \mu_1 \frac{\partial e}{\partial \mu_1}\bigg|_{bare} \frac{\partial Z_m^{-1}}{\partial e}$$

$$+ \mu_1 \frac{\partial c}{\partial \mu_1}\bigg|_{bare} \frac{\partial Z_m^{-1}}{\partial c}$$

$$= (-\lambda\varepsilon + \beta_1^{\lambda}) \frac{\partial}{\partial \lambda} Z_m^{-1} + \beta_1^{e} \frac{\partial}{\partial e} Z_m^{-1} + c \frac{\partial}{\partial c} Z_m^{-1} \qquad (5.18a)$$

$$\mu_2 \frac{\partial}{\partial \mu_2} Z_m^{-1} = \mu_2 \frac{\partial \lambda}{\partial \mu_2}\bigg|_{bare} \frac{\partial Z_m^{-1}}{\partial \lambda} + \mu_2 \frac{\partial e}{\partial \mu_2}\bigg|_{bare} \frac{\partial Z_m^{-1}}{\partial e}$$

$$+ \mu_2 \frac{\partial c}{\partial \mu_2}\bigg|_{bare} \frac{\partial Z_m^{-1}}{\partial c}$$

$$= \beta_2^{\lambda} \frac{\partial}{\partial \lambda} Z_m^{-1} + (-\frac{e\varepsilon}{2} + \beta_2^{e}) \frac{\partial}{\partial e} Z_m^{-1} - c \frac{\partial}{\partial c} Z_m^{-1} \qquad (5.18b)$$

We shall need numerical values of β_2 and $\gamma_m^{(1)}$:

$$\beta_2 = -3A_{11} = 3(\, 2\gamma_{m_1}^{(1)}\,) = \frac{3}{16\pi^2} \qquad (5.19)$$

We shall also need the following result proved by Collins [1]: If $H(\lambda, m, \varepsilon) Z_{m(0)}^{-1}(\lambda, \varepsilon)$ = finite at $\varepsilon = 0$, keeping λ and m fixed, and H is a finite function of λ and m at $\varepsilon = 0$, then

$$H(\lambda, m, \varepsilon) = 0 \qquad (5.20)$$

[5.3] IMPOVED ENERGY MOMENTUM TENSOR IN SCALAR QUANTUM ELECTRO DYNAMICS

We shall work with the following action,

$$S[\phi,g] = \int d^n x\sqrt{-g(x)}\Big[-\tfrac{1}{4}g^{\alpha\beta}(x)g^{\gamma\delta}(x)F_{\alpha\gamma}F_{\beta\delta} + \tfrac{1}{2}g^{\alpha\beta}(x)(D_\alpha\phi)^*(D_\beta\phi)$$
$$-m_o^2\phi^*\phi - \frac{\lambda_o}{4}(\phi^*\phi)^2 - \frac{1}{2}\xi_o\Big\{\frac{1}{\sqrt{-g(x)}}\partial_\alpha\{g^{\alpha\beta}(x)\sqrt{-g(x)}\;A_\beta(x)\}\Big\}^2\Big] \tag{5.21}$$

The energy momentum tensor derived from this action via Eq. (3.1) is given by the expression,

$$\theta_{\mu\nu} = -g_{\mu\nu}\mathcal{L} - F_{\mu\nu}F^{\mu\nu} + 2(D_\mu\phi)^*(D^\mu\phi) - \tfrac{1}{2}g_{\mu\nu}\xi(\partial.A)^2$$
$$-\frac{1}{2}g_{\mu\nu}\xi_o A^\rho\partial_\rho(\partial.A) + \frac{1}{2}\xi_o[A_\nu\partial_\mu(\partial.A) + A_\mu\partial_\nu(\partial.A)] \tag{5.22}$$

As has already been shown in Sec.[3.2], to prove the finiteness of $\theta_{\mu\nu}$, it is sufficient to prove the finiteness of its trace. $\theta_{\mu\nu}$ is obtained by standard manipulations,

$$\theta_\mu{}^\mu = (n-4)\mathcal{L} + 2\,(D_\mu\phi)^*\,(D^\mu\phi) - 4\,m_o^2\phi^*\phi - \lambda_o(\phi^*\phi)^2$$
$$+ (n-2)\xi_o\partial^\rho(A_\rho\partial.A) \tag{5.23}$$

$\theta_\mu{}^\mu$ in Eq. (5.23) does not have finite matrix elements beyond $O(\lambda)$ and an improvement term is necessarily needed to make it finite. As discussed in chapter 4, one may add to it any quantity whose divergence is zero and which does not contribute to the W.T.identity. The only such improvement term one needs to consider is of the form $(\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2)\phi^*\phi$. Therefore, we define the improved energy momentum tensor as

$$\theta^{imp}_{\mu\nu} = \theta_{\mu\nu} + \Big[\frac{n-2}{2(n-1)} + \frac{\tilde{g}}{1-n}\Big](\partial_\mu\partial_\nu - \partial^2 g_{\mu\nu})\phi^*\phi \tag{5.24}$$

where the improvement coeffic$\overset{e}{\wedge}$nt $\tilde{g}$ has been reparametrized for later convenience and we have the freedom to choose it. We will now investigate the possibility of choosing $\tilde{g}$ in such a manner that $\theta^{imp}_{\mu\nu}$ has finite matrix elements to all orders. It is sufficient to consider the finiteness of $\theta^{imp\mu}_{\mu}$, which is given by

$$\theta^{imp\mu}_{\mu} = (n-4)[-\frac{\lambda_o(\phi^*\phi)^2}{4} + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\xi_o(\partial.A)^2] + 2m_o^2\phi^*\phi + \\ - (n-2)\xi_o\partial^\rho(A_\rho\partial.A) - (\frac{n-2}{2})(\phi^*\frac{\delta S}{\delta\phi^*} + \frac{\delta S}{\delta\phi}\phi) \\ + \tilde{g}\ Z_m^{-1}\langle\partial^2(\phi^*\phi)\rangle \tag{5.25}$$

Using Eqs.(5.6),(5.8) and (5.9) and making use of the fact that $\xi_o\partial^\rho(A_\rho\partial.A)$ is a finite operator [see Appendix F for proof], we obtain the following expression for $\theta^{imp\mu}_{\mu}$,

$$\langle\theta^{imp\mu}_{\mu\nu}\rangle = \text{finite} + (n-4)\langle-\frac{\lambda_o(\phi^*\phi)^2}{4} + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\xi_o(\partial.A)^2\rangle^{UR} \\ + \tilde{g}\ Z_m^{-1}\langle\partial^2(\phi^*\phi)\rangle \tag{5.26}$$

At zero momentum the last term in Eq.(5.26) vanishes and we only need to consider finiteness of the operator

$$(n-4)\ \langle-\frac{\lambda_o(\phi^*\phi)^2}{4} + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\xi_o(\partial.A)^2\rangle$$

for considering the finiteness of $\langle\theta^{imp\mu}_{\mu}\rangle$. One can use the technique of Ref.8 to obtain

$$(n-4)\ \langle\int(-\frac{\lambda_o(\phi^*\phi)^2}{4} + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\xi_o(\partial.A)^2)d^nx\ \rangle^{U.R}$$

We shall do so in the next section to show that $\langle\theta^{imp\mu}_{\mu}\rangle$ is finite at zero momentum and to obtain an expression for the trace anamoly at zero momentum. This will also furnish information about the renormalization matrix Z_{ij} of Eq.(5.15) which will be used in Sec.[5.5] for obtaining $\langle\theta^{imp\mu}_{\mu}\rangle^{U.R}$ at non

zero momentum.

[5.4]FINITENESS OF $\theta_{\mu}^{\iota mp\mu}$ AT ZERO MOMENTUM

At zero momentum, the trace equation is

$$\langle \int \theta_{\mu}^{\iota mp\mu} d^n x\rangle^{U.R} = (n-4)\langle \int (-\frac{\lambda_o(\phi^*\phi)^2}{4} + \frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\xi_o(\partial.A)^2)d^n x\rangle^{U.R}$$
$$= \text{finite} \qquad (5.27)$$

It is straightforward to show that

$$\langle \int (-\frac{\lambda_o(\phi^*\phi)^2}{4} + \frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\xi_o(\partial.A)^2)d^n x\rangle^{U.R}$$
$$= -\langle S\rangle^{U.R} + \frac{1}{2}\langle \int (\phi^* \frac{\delta S}{\delta\phi^*} + \frac{\delta S}{\delta\phi}\phi)d^n x\rangle^{R} \qquad (5.28)$$

To obtain $\langle S\rangle^{U.R}$ we proceed as follows [8] . The generating functional of regularized Green's function is defined as

$$W[\lambda_o a, e_o\sqrt{a}, m_o^2, \sqrt{a}J, \sqrt{a}J, \sqrt{a}J^*, \sqrt{a}J_{\mu}, \xi_o, \varepsilon]$$
$$= \int \mathcal{D}\phi\mathcal{D}\phi^*\mathcal{D}A_{\mu}\exp(\frac{\iota}{a}S + \int d^n x[J^*\phi + J\phi^* + J_{\mu}A^{\mu}] \qquad (5.29)$$

which implies that

$$-\iota\langle S\rangle W = \left.\frac{\partial W}{\partial a}\right|_{a=1}$$
$$= \lambda_o\frac{\partial W}{\partial\lambda_o} + \frac{1}{2}e_o\frac{\partial W}{\partial e_o} + \frac{1}{2}\int d^n x\, J(x)\frac{\delta W}{\delta J(x)} + \frac{1}{2}\int d^n x\, J^*(x)\frac{\delta W}{\delta J^*(x)}$$
$$+ \frac{1}{2}\int d^n x\, J_{\mu}(x)\frac{\delta W}{\delta J_{\mu}(x)} \qquad (5.30)$$

Now ,

$$\lambda_o\frac{\partial W}{\partial\lambda_o} + \frac{1}{2}e_o\frac{\partial W}{\partial e_o}$$
$$= \lambda_o\left.\frac{\partial W^R}{\partial\lambda_o}\right|_{bare} + \frac{1}{2}e_o\left.\frac{\partial W}{\partial e_o}\right|_{bare}$$

where $W = W^R = W^R[\mathcal{J}^R, \lambda, e, m^2, \xi, \mu_1, \mu_2, c, \varepsilon]$

The above expression is equal to

$$(\lambda_o\frac{\partial\lambda}{\partial\lambda_o} + \frac{1}{2}e_o\frac{\partial\lambda}{\partial e_o})\frac{\partial W^R}{\partial\lambda} + (\lambda_o\frac{\partial e}{\partial\lambda_o} + \frac{1}{2}e_o\frac{\partial e}{\partial e_o})\frac{\partial W^R}{\partial e} + (\lambda_o\frac{\partial m^2}{\partial\lambda_o} + \frac{1}{2}e_o\frac{\partial m^2}{\partial e_o})\frac{\partial W^R}{\partial m^2}$$

$$+\int d^n x\ \frac{\delta W^R}{\delta J^R}\left(\lambda_o\frac{\partial J^R}{\partial\lambda_o}+\frac{1}{2}e_o\frac{\partial J^R}{\partial e_o}\right)+\int d^n x\ \frac{\delta W^R}{\delta J^{*R}}\left(\lambda_o\frac{\partial J^{*R}}{\partial\lambda_o}+\frac{1}{2}e_o\frac{\partial J^{*R}}{\partial e_o}\right)$$

$$+\int d^n x\ \frac{\delta W^R}{\delta J^R_\mu}\left(\lambda_o\frac{\partial J^R_\mu}{\partial\lambda_o}+\frac{1}{2}e_o\frac{\partial J^R_\mu}{\partial e_o}\right) \qquad (5\ 31)$$

Now λ depends on λ_o and e_o through the combination $\lambda_o\mu_1^{-\varepsilon}$ and $e_o\mu_2^{-\varepsilon/2}$.

Hence,

$$\left.\left(\lambda_o\frac{\partial\lambda}{\partial\lambda_o}+\frac{1}{2}e_o\frac{\partial\lambda}{\partial e_o}\right)\right|_{bare}=-\frac{1}{\varepsilon}\left(\mu_1\frac{\partial\lambda}{\partial\mu_1}+\mu_2\frac{\partial\lambda}{\partial\mu_2}\right)$$

$$=-\frac{1}{\varepsilon}[\beta_1^\lambda(\lambda,e,c,\varepsilon)+\beta_2^\lambda(\lambda,e,c,\varepsilon)]$$

with similar results holding for all other terms. Now using definitions of Eq.(5.10), Eq.(5.30) reduces to,

$$-\iota\langle S\rangle^{U.R}\ W^R=-\left[\frac{\beta_1^\lambda(\lambda,e,c,\varepsilon)+\beta_2^\lambda(\lambda,e,c,\varepsilon)}{\lambda\varepsilon}\right]\lambda\ \frac{\partial W^R}{\partial\lambda}$$

$$-\left[\frac{\beta_1^e(\lambda,e,c,\varepsilon)+\beta_2^e(\lambda,e,c,\varepsilon)}{e\varepsilon}\right]e\ \frac{\partial W^R}{\partial e}$$

$$-\left[\frac{\gamma_{m_1}+\gamma_{m_2}}{\varepsilon}\right]2m^2\ \frac{\partial W^R}{\partial m^2}-\left[\frac{\gamma_{\xi_1}+\gamma_{\xi_2}}{\varepsilon}\right]\xi\ \frac{\partial W^R}{\partial\xi}$$

$$+\left[\frac{1}{2}-\frac{\gamma_1}{\varepsilon}-\frac{\gamma_2}{\varepsilon}\right]\int d^n x\left[J^R\ \frac{\delta W^R}{\delta J^R}+J^{*R}\ \frac{\delta W^R}{\delta J^{*R}}\right]$$

$$+\left[\frac{1}{2}-\frac{\gamma_1'}{\varepsilon}-\frac{\gamma_2'}{\varepsilon}\right]\int d^n x\ J^R_\mu\ \frac{\delta W^R}{\delta J^R_\mu} \qquad (5.33)$$

Substituting the above in Eq.(5.27) and using the definitions (5.13), one obtains

$$\langle\int O_1 d^n x\rangle^{U.R}=\sum_{j=1}^{\sigma}Z_{1j}\langle\int X_j d^n x\rangle^R \qquad (5.34)$$

where

$$Z_{11} = 1 - \frac{\beta_1^\lambda + \beta_2^\lambda}{\lambda\varepsilon}$$

$$Z_{12} = 2\frac{\gamma_{m_1} + \gamma_{m_2}}{\varepsilon}$$

$$Z_{13} = \frac{\gamma_1 + \gamma_2}{\varepsilon}$$

$$Z_{14} = -\frac{\beta_1^e + \beta_2^e + \gamma_1' + \gamma_2'}{\varepsilon}$$

$$Z_{15} = \left[\frac{\beta_1^\lambda + \beta_2^\lambda}{\lambda\varepsilon} - 2\frac{\beta_1^e + \beta_2^e}{e\varepsilon}\right]$$

$$Z_{16} = \frac{\gamma_{\xi_1} + \gamma_{\xi_2}}{\varepsilon} \tag{5.35}$$

Note that Z_{1j}, $j = 1,2,\cdots\cdots,6$ have only simple poles in ε and hence

$$\langle \int d^n x\, \theta_\mu^{\mu imp} \rangle = \text{finite} \tag{5.36}$$

[5.5] θ_μ^μ AT NON-ZERO MOMENTUM : IMPROVEMENT TERM DEPENDENCE OF THE FORM $\tilde{g}(\varepsilon, e_0^2\mu_2^{-\varepsilon}, \lambda_0\mu_1^{-\varepsilon}, c)$

We have shown in the previous section that the energy momentum tensor

$$\theta_{\mu\nu}^{imp'} = \theta_{\mu\nu} + \frac{n-2}{2(1-n)}(\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2)\phi^*\phi$$

is finite at zero momentum. At non-zero momentum, one can show, by explicit calculations, that this energy momentum tensor is finite only upto $O(\lambda^3)$ at e=0, upto $O(e^4)$ at $\lambda = 0$ and also in $O(\lambda e^2)$, but a further improvement is <u>necessarily</u> needed in $O(\lambda^4)$, $O(\lambda e^4)$ and $O(\lambda^2 e^2)$ [See for exapmle Appendix E].

We shall now consider a further improvement, where the improvement coefficient is a finite function of <u>bare</u> coupling

constants, ε and $c(=\mu_1/\mu_2)$:

$$\theta^{imp\,'}_{\mu\nu} = \theta_{\mu\nu} + \left[\frac{n-2}{2(1-n)} + \frac{\tilde{g}(\varepsilon, e_0^2\mu_2^{-\varepsilon}, \lambda_0\mu_1^{-\varepsilon}, c)}{1-n}\right](\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2)\phi^*\phi \tag{5.37}$$

$$\tilde{g}(\varepsilon, e_0^2\mu_2^{-\varepsilon}, \lambda_0\mu_1^{-\varepsilon}, c) = \sum_{r=0}^{\infty} \tilde{g}_r(\varepsilon, \lambda_0\mu_1^{-\varepsilon}, c)(e_0^2\mu_2^{-\varepsilon})^r \tag{5.38}$$

and $\tilde{g}_r$'s are finite functions of ε and $\lambda_0\mu_1^{-\varepsilon}$.

Trace of $\theta^{imp}_{\mu\nu}$ is, as before,

$$\langle\theta^{imp\,\mu}_{\mu}\rangle = [\tilde{g}(\varepsilon, e_0^2\mu_2^{-\varepsilon}, \lambda_0\mu_1^{-\varepsilon}, c)Z_m^{-1} - \varepsilon Z_{17}]\langle\partial^2(\phi^*\phi)\rangle^R + \text{finite} \tag{5.39}$$

One can reexpress for future convenience,

$$\tilde{g}(\varepsilon, e_0^2\mu_2^{-\varepsilon}, \lambda_0\mu_1^{-\varepsilon}, c) = -\varepsilon g(\varepsilon, e_0^2\mu_2^{-\varepsilon}, \lambda_0\mu_1^{-\varepsilon}, c) \tag{5.40}$$

$$\tilde{g}_n(\varepsilon, , \lambda_0\mu_1^{-\varepsilon}, c) = -\varepsilon g_n(\varepsilon, , \lambda_0\mu_1^{-\varepsilon}, c) \tag{5.41}$$

where $g(\varepsilon, e_0^2\mu_2^{-\varepsilon}, \lambda_0\mu_1^{-\varepsilon}, c)Z_m^{-1}$ and $g_{m's}$ are allowed to have $1/\varepsilon$ terms. We thus have

$$\langle\theta^{imp\,\mu}_{\mu}\rangle = \text{finite} - \varepsilon X\langle\partial^2(\phi^*\phi)\rangle^R \tag{5.42}$$

where $$X = Z_{17} + g(\varepsilon, e_0^2\mu_2^{-\varepsilon}, \lambda_0\mu_1^{-\varepsilon}, c)Z_m^{-1} \tag{5.43}$$

Thus, to obtain a finite energy momentum tensor, one must find a $g(\varepsilon, e_0^2\mu_2^{-\varepsilon}, \lambda_0\mu_1^{-\varepsilon}, c)$ such that X, given by Eq.(5.43) has no worse than simple poles. We shall show, in what follows, that it is not possible to do so consistently except at e = 0. To achieve this end, we shall make use of RG equations satisfied by Z_{17} [See Appendix G]:

$$(-\lambda\varepsilon + \beta_1^\lambda + \beta_2^\lambda)\frac{\partial Z_{17}}{\partial\lambda} + (-\frac{e\varepsilon}{2} + \beta_1^e + \beta_2^e)\frac{\partial Z_{17}}{\partial e} - 2(\gamma_{m_1} + \gamma_{m_2})\, Z_{17}$$

$$= \quad Z_{11}(\gamma_{17}+\gamma'_{17}) + Z_{15}\;(\gamma_{57}+\gamma'_{57}) \qquad (G.11)$$

Substituting for Z_{17} from Eq.(5.43) and using

$$\mu_1\frac{\partial}{\partial\mu_1}[g(\varepsilon,e_o^2\mu_2^{-\varepsilon},\lambda_o\mu_1^{-\varepsilon},c)Z_m^{-1}] = 2g(\varepsilon,e_o^2\mu_2^{-\varepsilon},\lambda_o\mu_1^{-\varepsilon},c)\gamma_{m_1}Z_m^{-1} + Z_m^{-1}\mu_1\frac{\partial}{\partial\mu_1}g(\varepsilon,e_o^2\mu_2^{-\varepsilon},\lambda_o\mu_1^{-\varepsilon},c) \qquad (5.44a)$$

$$\mu_2\frac{\partial}{\partial\mu_2}[g(\varepsilon,e_o^2\mu_2^{-\varepsilon},\lambda_o\mu_1^{-\varepsilon},c)Z_m^{-1}] = 2g(\varepsilon,e_o^2\mu_2^{-\varepsilon},\lambda_o\mu_1^{-\varepsilon},c)\gamma_{m_2}Z_m^{-1} + Z_m^{-1}\mu_2\frac{\partial}{\partial\mu_2}g(\varepsilon,e_o^2\mu_2^{-\varepsilon},\lambda_o\mu_1^{-\varepsilon},c) \qquad (5.44b)$$

and Eq.(G.11) one obtains the following equation satisfied by X

$$(-\lambda\varepsilon + \beta_1^\lambda + \beta_2^\lambda)\frac{\partial X}{\partial\lambda} + (-\frac{e\varepsilon}{2} + \beta_1^e + \beta_2^e)\frac{\partial X}{\partial e} - 2(\gamma_{m_1}+\gamma_{m_2})X$$
$$= Z_{11}(\gamma_{17}+\gamma'_{17}) + Z_{15}(\gamma_{57}+\gamma'_{57}) - [(\mu_1\frac{\partial}{\partial\mu_1}+\mu_2\frac{\partial}{\partial\mu_2})g(\varepsilon,e_o^2\mu_2^{-\varepsilon},\lambda_o\mu_1^{-\varepsilon},c)]Z_m^{-1} \qquad (5.45)$$

which together with

$$g(\varepsilon,e_o^2\mu_2^{-\varepsilon},\lambda_o\mu_1^{-\varepsilon},c) = \sum_{n=0}^{\infty} g_n(\varepsilon,\lambda_o\mu_1^{-\varepsilon},c)(e_o^2\mu_2^{-\varepsilon})^n \qquad (5.46)$$

reduces to

$$(-\lambda\varepsilon + \beta_1^\lambda + \beta_2^\lambda)\frac{\partial X}{\partial\lambda} + (-\frac{e\varepsilon}{2} + \beta_1^e + \beta_2^e)\frac{\partial X}{\partial e} - 2(\gamma_{m_1}+\gamma_{m_2})X$$
$$= Z_{11}(\gamma_{17}+\gamma'_{17}) + Z_{15}(\gamma_{57}+\gamma'_{57}) + \varepsilon\left[\frac{\partial g_o}{\partial(\lambda_o\mu_1^{-\varepsilon})}\lambda_o\mu_1^{-\varepsilon} + \sum_{n=1}^{\infty}(e_o^2\mu_2^{-\varepsilon})^n\left\{n\,g_n + \frac{\partial g_n}{\partial(\lambda_o\mu_1^{-\varepsilon})}\lambda_o\mu_1^{-\varepsilon}\right\}\right] \qquad (5.47)$$

It should be noted that this equation is always satisfied by X for any $g(\varepsilon,e_o^2\mu_2^{-\varepsilon},\lambda_o\mu_1^{-\varepsilon},c)$ (that is for any g_n's). We shall consider this equation at c = 1, as this suffices for our purpose and we will write, from now onwards,

$$g(\varepsilon, e_o^2\mu_2^{-\varepsilon}, \lambda_o\mu_1^{-\varepsilon}) \equiv g(\varepsilon, e_o^2\mu_2^{-\varepsilon}, \lambda_o\mu_1^{-\varepsilon}, c)\,\big|_{c=1}$$

$$g_n(\varepsilon, \lambda_o\mu_1^{-\varepsilon}) = g_n(\varepsilon, \lambda_o\mu_1^{-\varepsilon}, c)\big|_{c=1}$$

Now suppose, if possible, that g_n's can be chosen such that X has no worse than simple poles (which implies the existence of a finite e.m.tensor). Then the l.h.s. of Eq.(5.47) contains at worst simple poles. Also Z_{11} and Z_{15} have only simple poles [see Eq.(5.35)] and γ_{17}, γ_{17}', γ_{57} and γ_{57}' are finite quantities. Therefore

$$\varepsilon\left[\frac{\partial g_o}{\partial(\lambda_o\mu_1^{-\varepsilon})}\lambda_o\mu_1^{-\varepsilon} + \sum_{n=o}^{\infty}(e_o^2\mu_2^{-\varepsilon})^n\left\{ng_n + \frac{\partial g_n}{\partial(\lambda_o\mu_1^{-\varepsilon})}\lambda_o\mu_1^{-\varepsilon}\right\}\right] \tag{5.48}$$

also has at worst simple poles. Thus, we should have, for the finiteness of $\theta_{\mu\nu}^{imp}$,

$$\varepsilon^2\left[\frac{\partial g_o}{\partial(\lambda_o\mu_1^{-\varepsilon})}\lambda_o\mu_1^{-\varepsilon} + \sum_{n=1}^{\infty}(e_o^2\mu_2^{-\varepsilon})^n\left\{ng_n + \frac{\partial g_n}{\partial(\lambda_o\mu_1^{-\varepsilon})}\lambda_o\mu_1^{-\varepsilon}\right\}\right] Z_m^{-1}$$
$$= \text{finite} \tag{5.49}$$

At e = 0, Eq.(5.49) reduces to

$$\varepsilon^2\left[\frac{\partial g_o}{\partial(\lambda_o\mu_1^{-\varepsilon})}\lambda_o\mu_1^{-\varepsilon}\right] Z_{m(o)}^{-1} = \text{finite} \tag{5.50}$$

which, using the Uniqueness Theorem of chapter 4, implies that

$$\varepsilon^2\left[\frac{\partial g_o}{\partial(\lambda_o\mu_1^{-\varepsilon})}\lambda_o\mu_1^{-\varepsilon}\right] = 0 \tag{5.51}$$

i.e. g_o is a function of ε only. This is in agreement with the result of Ref.1.

Using Eq.(5.51), Eq.(5.49) in $O(e^2)$ implies that

$$\varepsilon^2\left[g_1 + \frac{\partial g_1}{\partial(\lambda_o\mu_1^{-\varepsilon})}\lambda_o\mu_1^{-\varepsilon}\right] Z_{m(o)}^{-1} = \text{finite} \tag{5.52}$$

Therefore, as before, one obtains

$$\varepsilon^2\left[g_1(\varepsilon,\lambda_o\mu_1^{-\varepsilon}) + \frac{\partial g_1}{\partial(\lambda_o\mu_1^{-\varepsilon})}\lambda_o\mu_1^{-\varepsilon}\right] = 0 \tag{5.53}$$

Now, εg_1 is a finite function of $\lambda_o\mu_1^{-\varepsilon}$, therefore it can be expanded as

$$\varepsilon g_1(\varepsilon,\lambda_o\mu_1^{-\varepsilon}) = \sum_{n=0}^{\infty} \varepsilon g_1^{(n)}(\varepsilon)(\lambda_o\mu_1^{-\varepsilon})^n \tag{5.54}$$

where $\varepsilon g_1^{(n)}(\varepsilon)$ is finite at $\varepsilon = 0$.

Substituting Eq.(5.54) in Eq.(5.53), one obtains

$$[g_1^{(0)}(\varepsilon)+\sum_{n=0}^{\infty}(g_1^{(n)}(\varepsilon) + ng_1^{(n)}(\varepsilon))(\lambda_o\mu_1^{-\varepsilon})^n] = 0 \tag{5.55}$$

Comparing powers of λ_o on both sides of Eq.(5.55) one staightforwardly obtains

$$g_1^{(n)} = 0 \qquad n = 0,\ 1,\ 2,\ 3,\ \cdots\cdots \tag{5.56}$$

Eq.(5.56) means that to $O(e^2)$ and all orders in λ no additional improvement is needed to make $\theta_\mu^{\ \mu}$ finite. But this contadicts the result of Appendix E, where we have shown that in $O(\lambda^2 e^2)$, the quantity $[-\varepsilon Z_{17} + \tilde{g}_0(\varepsilon)Z_m^{-1}]$ <u>does</u> have double poles. Hence we conclude that it is not possible to find an improved energy momentum tensor of the form given in Eq.(5.37), which may be finite even to $O(e^2)$.

[5.6] $\theta_\mu^{\ \mu}$ AT NON-ZERO MOMENTUM : IMPROVEMENT TERM DEPENDENCE OF THE FORM $\tilde{g}(\varepsilon,\ e^2,\ \lambda,\ c)$

We shall now consider an improved energy momentum tensor of the form

$$\theta_{\mu\nu}^{imp} = \theta_{\mu\nu} + \left[\frac{n-2}{2(1-n)} + \frac{\tilde{g}(\varepsilon,e^2,\lambda,c)}{1-n}\right](\partial_\mu\partial_\nu-\partial^2 g_{\mu\nu})\phi^*\phi \tag{5.57}$$

Retracing the steps of the previous section, we obtain the

trace of $\theta^{\iota mp}_{\mu\nu}$:

$$\langle\theta_{\mu}^{\iota mp\mu}\rangle = \text{finite} - \varepsilon X\langle\partial^2(\phi^*\phi)\rangle^R \tag{5.58}$$

where

$$X = Z_{17} + g(\varepsilon, e^2, \lambda, c) Z_m^{-1}$$

and

$$g(\varepsilon, e^2, \lambda, c) \equiv \sum_{n=0}^{\infty} e^{2n} g_n(\varepsilon, \lambda, c) = -\frac{1}{\varepsilon}\tilde{g}(\varepsilon, e^2, \lambda, c) \tag{5.59}$$

X satisfies the following equation

$$\begin{aligned}(-\lambda\varepsilon + \beta_1^\lambda + \beta_2^\lambda)\frac{\partial X}{\partial\lambda} + (-\frac{e\varepsilon}{2} + \beta_1^e + \beta_2^e)\frac{\partial X}{\partial e} - 2(\gamma_{m_1} + \gamma_{m_2})X \\ = Z_{11}(\gamma_{17} + \gamma_{17}') + Z_{15}(\gamma_{57} + \gamma_{57}') \\ -\left[\mu_1\frac{\partial}{\partial\mu_1}g(\varepsilon, e^2, \lambda, c) + \mu_2\frac{\partial}{\partial\mu_2}g(\varepsilon, e^2, \lambda, c)\right]Z_m^{-1}\end{aligned} \tag{5.60}$$

Now,

$$\mu_1\frac{\partial}{\partial\mu_1}g(\varepsilon, e^2, \lambda, c) = (-\lambda\varepsilon + \beta_1^\lambda)\frac{\partial g}{\partial\lambda} + \beta_1^e\frac{\partial g}{\partial e} + c\frac{\partial g}{\partial c} \tag{5.61a}$$

$$\mu_2\frac{\partial}{\partial\mu_2}g(\varepsilon, e^2, \lambda, c) = (-\frac{e\varepsilon}{2} + \beta_2^e)\frac{\partial g}{\partial e} + \beta_2^\lambda\frac{\partial g}{\partial\lambda} - c\frac{\partial g}{\partial c} \tag{5.61b}$$

Using Eqs.(5.61a) and (5.61b), Eq (5.60) reduces to

$$\begin{aligned}(-\lambda\varepsilon + \beta_1^\lambda + \beta_2^\lambda)\frac{\partial X}{\partial\lambda} + (-\frac{e\varepsilon}{2} + \beta_1^e + \beta_2^e)\frac{\partial X}{\partial e} - 2(\gamma_{m_1} + \gamma_{m_2})X \\ = Z_{11}(\gamma_{17} + \gamma_{17}') + Z_{15}(\gamma_{57} + \gamma_{57}') \\ -\left[(-\lambda\varepsilon + \beta_1^\lambda + \beta_2^\lambda)\frac{\partial g}{\partial\lambda} + (-\frac{e\varepsilon}{2} + \beta_1^e + \beta_2^e)\frac{\partial g}{\partial e}\right]Z_m^{-1}\end{aligned} \tag{5.62}$$

As before, it will prove sufficient to consider this equation at c=1. If X has no worse than simple poles then, as argued in Sec. [5.5], Eq.(5.62) implies that

$$\varepsilon\left[(-\lambda\varepsilon + \beta_1^\lambda + \beta_2^\lambda)\frac{\partial g}{\partial\lambda} + (-\frac{e\varepsilon}{2} + \beta_1^e + \beta_2^e)\frac{\partial g}{\partial e}\right]Z_m^{-1} = \text{finite} \tag{5.63}$$

Substituting the expression of Eq.(5.59), Eq.(5.63) reduces to

$$\varepsilon\left[(-\lambda\varepsilon+\beta_1^{\lambda}+\beta_2^{\lambda})\sum_{n=0}^{\infty} e^{2n}\frac{\partial g_n}{\partial\lambda} +(-\frac{e\varepsilon}{2}+\beta_1^{e}+\beta_2^{e})\sum_{n=1}^{\infty} 2ne^{2n-1}g_n\right]Z_m^{-1}$$

$$= \text{finite} \qquad (5.64)$$

At e=0, Eq.(5.64) reduces to

$$\varepsilon\left[(-\lambda\varepsilon + \beta_2\lambda^2 + \beta_3\lambda^3 + \cdots\cdots)\frac{\partial g_0}{\partial\lambda}\right]Z_{m(0)}^{-1} = \text{finite} \qquad (5.65)$$

Here, $g_0(\varepsilon)$ is a finite function of λ and ε, therefore using the result in Ref.1 [see Eq.(5.20)], one obtains

$$\varepsilon(-\lambda\varepsilon + \beta_2\lambda^2 + \beta_3\lambda^3 + \cdots\cdots)\frac{\partial g_0}{\partial\lambda} = 0 \qquad (5.66)$$

But $(-\lambda\varepsilon + \beta_2\lambda^2 + \beta_3\lambda^3 + \cdots\cdots) \neq 0$, therefore

$$\varepsilon\ g_0(\lambda,\varepsilon) \equiv \varepsilon\ g_0(\varepsilon)$$

i.e. g_0 is a finite function of ε only. This again agrees with the previous result [1].

In $O(e^2)$ Eq.(5.64) reduces to

$$\varepsilon\left[(-\lambda\varepsilon + \beta_2\lambda^2 + \beta_3\lambda^3 + \cdots\cdots)\frac{\partial g_1}{\partial\lambda} - \varepsilon\ g_1(\lambda,\varepsilon)\right]Z_{m(0)}^{-1} = \text{finite}$$

$$(5.67)$$

which, as before, implies that

$$(-\lambda\varepsilon + \beta_2\lambda^2 + \beta_3\lambda^3 + \cdots\cdots)\frac{\partial g_1}{\partial\lambda} - \varepsilon\ g_1(\lambda,\varepsilon) = 0 \qquad (5.68)$$

Expanding

$$g_1(\lambda,\varepsilon) = \sum_{n=0}^{\infty} g_1^{(n)}(\varepsilon)\lambda^n \qquad (5.69)$$

and comparing successive powers of λ in Eq.(5.68) one finally obtains

$$g_1^{(n)} = 0 \text{ for all } n \qquad (5.70)$$

Hence if an improvement of the form given in Eq.(5.57) is sufficient to make $\theta_\mu^{\mu imp}$ finite, then one need not have any

improvement in $O(e^2)$ and to all orders in λ. But this contradicts the result in Appendix E and hence this improvement program also fails in $O(e^2)$. [It is easy to see that the above analysis can be extended to all orders in e^2, suggesting thereby that the improvement coefficient $g_o(\varepsilon)$, as obtained in Ref.1 should be sufficient to all orders, if an improvement program of the form in Eq.(5.57) is to be successful. But this obviously is not true].

[5.7]CONCLUSION

We have considered the possibility of improving the energy momentum tensor of scalar quantum electrodynamics, in the framework of dimensional regularization, by adding an improvement coefficient depending on ε. A finite improvement coefficient depending on ε is not sufficient except at e=0. The most desirable choice of a more general improvement term would be one in which the improvement coefficient is a finite function of ε and bare coupling constants, because such an improved energy momentum tensor would be derivable from bare action. But, it was found that this type of improvement coefficient is not sufficient even in $O(\lambda^2 e^2)$. Moreover, an improvement term, which is a finite function of ε and renormalized coupling constants, was also not found to work beyond same order. It is , therefore necessary to introduce a new infinite renormalization in order to get finite matrix elements for energy momentum tensor in scalar electrodynamics.

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CHAPTER 6

ENERGY MOMENTUM TENSOR IN THEORIES WITH SCALAR FIELDS AND TWO COUPLING CONSTANTS

[6.1]INTRODUCTION

It was shown in the previous chapter that the energy momentum tensor in scalar Quantum Electrodynamics is not finite and that it is impossible to make it finite by adding improvement terms consistent with finite improvement program[1]. Now, we will analyze three more theories involving scalar fields and having two coupling constants. These theories are

(i) Non- abelian gauge theories with scalars(NAGT's with scalars)

(ii) Yukawa theory

(iii) A theory involving two interacting scalar fields.

We shall establish, in the context of each of these theories, the impossibility of having a finite improvement program leading to a finite energy momentum tensor[2-4]. As before, we shall consider improved energy momentum tensors of two kinds:

[A] When the improvement coefficient is a finite function of bare coupling constant : $g \equiv g(\varepsilon, \lambda_o \mu^{-\varepsilon}, \varkappa_o \mu^{-\varepsilon})$.

[B] When the improvement coefficient is a finite function of renormalized coupling constants: $g \equiv g(\varepsilon, \lambda, \varkappa)$.

The analysis for all the three cases is very similar to the previous case. For this reason, we shall be very brief.

[6.2]ENERGY MOMENTUM TENSOR IN NAGT'S WITH SCALARS

We shall consider a real scalar multiplet in the vector representation of O(3) coupled to O(3) gauge fields. (Generalization to O(N) group is straightforward). The Lagrangian density is

$$L = L_o + L_g + L_{gh}; \qquad S = \int d^n x\, L \tag{6.1}$$

where

$$\begin{aligned} L_o &= -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{1}{2}(D_\mu \phi)^T (D^\mu \phi) - \frac{1}{2} m_o^2 \phi^2 - \frac{\lambda_o}{8}(\phi^T \phi) \\ L_g &= -\frac{1}{2} \sum_a \xi_o (\partial . A^a)^2 \\ L_{gh} &= \partial^\mu \overline{C}^a D^{ab}_\mu C_b \end{aligned} \tag{6.2}$$

and

$$D_\mu \phi = (\partial_\mu - \iota e_o T^a A^a_\mu)\phi \ ; \ F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + e_o f^{abc} A^b_\mu A^{c\nu}$$

T^a being the adjoint representation of O(3) satisfying

$$[T^a, T^b] = \iota f^{abc} T^c$$

The energy momentum tensor obtained from this action via Eq.(3.1) is

$$\begin{aligned} \theta_{\mu\nu} = &- g_{\mu\nu} L - F^a_{\mu\alpha} F^{a\alpha}_\nu + \frac{1}{2}[(D_\mu \phi)^T (D_\nu \phi) + (D_\nu \phi)^T (D_\mu \phi)] - \partial_\mu \overline{C}^a D^{ab}_\nu C_b \\ &- \partial_\nu \overline{C}^a D^{ab}_\mu C_b - g_{\mu\nu} \xi_o (\partial . A^a)^2 + \xi_o \partial_\mu (\partial . A^a) A^a_\nu + \xi_o \partial_\nu (\partial . A^a) A^a_\mu \\ &- g_{\mu\nu} \xi_o \partial^\rho (\partial . A^a) A^a_\rho \end{aligned} \tag{6.3}$$

This energy momentum tensor has finite matrix elements at q=0 and to first order in q[8], but not to second order in q. The most general improvement one can add to $\theta_{\mu\nu}$ is parametrized as

$$\theta^{imp}_{\mu\nu} = \theta_{\mu\nu} + \left[\frac{n-2}{4(1-n)} + \frac{\tilde{g}}{1-n}\right](\partial_\mu \partial_\nu - g_{\mu\nu}\partial^2)(\phi^T \phi)$$

$$\equiv \tilde{\theta}_{\mu\nu} + \frac{\tilde{g}}{1-n}(\partial_\mu \partial_\nu - g_{\mu\nu}\partial^2)(\phi^T\phi) \qquad (6.4)$$

where $\tilde{\theta}_{\mu\nu}$ is the energy momentum tensor obtained from the conformally invariant action[7].

To prove the finiteness of $\theta^{imp}_{\mu\nu}$, it is sufficient to prove the finiteness of $\theta_\mu{}^{imp\mu}$. It is straightforward to show that

$$\langle\theta_\mu{}^{imp\mu}\rangle = (n-4)\left\langle\left[-\frac{\lambda_o}{8}(\phi^T\phi)^2 + \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2}\xi_o(\partial.A^a)^2\right]\right\rangle^{U.R.}$$

$$-(n-2)\langle\bar{C}\frac{\delta S}{\delta\bar{C}}\rangle^{U.R.} - \left(\frac{n-2}{2}\right)\langle\phi_i\frac{\delta S}{\delta\phi_i}\rangle^{U.R.}$$

$$-(n-2)\langle\partial^\mu[\xi_o(\partial.A^a)A^a_\mu - \bar{C}^a D^{ab}_\mu C_b]\rangle^{U.R.}$$

$$+\langle m_o^2\phi^T\phi\rangle^{U.R.} + \tilde{g}\,\langle\partial^2(\phi^T\phi)^2\rangle^{U.R.} \qquad (6.5)$$

As shown in Appendix H, $\partial^\mu[\xi_o(\partial.A^a)A^a_\mu - \bar{C}^a D^{ab}_\mu C_b]$ is a finite operator. Moreover, $\bar{C}\frac{\delta S}{\delta\bar{C}}$,$\phi_i\frac{\delta S}{\delta\phi_i}$ and $m_o^2\phi^T\phi$ are also finite operators[This can be shown along the same lines as in Eqs.(5.6) and (4.13)]. Thus

$$\langle\theta_\mu{}^{imp\mu}\rangle = \text{finite} + (n-4)\langle O_1\rangle^{U.R.} + \tilde{g}\,Z_m^{-1}\langle\partial^2(\phi^T\phi)\rangle^{R} \qquad (6.6)$$

where

$$O_1 \equiv -\frac{\lambda_o}{8}(\phi^T\phi)^2 + \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2}\xi_o(\partial.A^a)^2$$

and we have used

$$\langle\partial^2(\phi^T\phi)^2\rangle^{U.R.} = Z_m^{-1}\langle\partial^2(\phi^T\phi)^2\rangle^{R}$$

[This can be shown analogously to Eq.(4.15)]. Thus, we only need to know the renormalization of O_1. O_1 can mix under renormalization with the following complete set of operators:

$$O_1 = -\frac{\lambda_o}{8}(\phi^T\phi)^2 + \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2}\xi_o(\partial.A^a)^2$$

$$O_2 = m_o^2 \phi^T \phi$$

$$O_3 = \phi_\iota \frac{\delta S}{\delta \phi_\iota} \quad ; \quad O_4 = A_\mu^a \frac{\delta \tilde{S}}{\delta A_\mu^a} + \partial_\mu \overline{C}^a D^{\mu ab} C_b$$

$$\text{where } \tilde{S} = S_o + \frac{1}{2}\int d^n x \xi_o \sum_a (\partial . A^a)^2$$

$$O_5 = \frac{\delta S}{\delta C^a} C^a$$

$$O_6 = \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{1}{2}\xi_o (\partial . A^a)^2$$

$$O_7 = \frac{1}{2} \xi_o (\partial . A^a)^2$$

$$O_8 = \partial^2 (\phi^T \phi) \tag{6.7}$$

One defines the renormalization matrix by

$$\langle O_\iota \rangle^{U.R} = \sum_{\iota=1}^{8} Z_{\iota j} \langle O_j \rangle^R \tag{6.8}$$

One can define the renormalized operators at zero momentum by the following equations:

$$\langle \int d^n x \; O_1^R \rangle = \lambda \frac{\partial Z^R}{\partial \lambda} + \frac{1}{2} e \frac{\partial Z^R}{\partial e} + \int d^n x \; J_\mu^R \frac{\delta Z^R}{\delta J_\mu^R}$$

$$\langle \int d^n x \; O_2^R \rangle = -2m^2 \frac{\partial Z^R}{\partial m^2}$$

$$\langle \int d^n x O_3^R \rangle = -\int d^n x J_\mu^R \frac{\delta Z^R}{\delta J_\mu^R} = \int d^n x O_3$$

$$\langle \int d^n x O_4^R \rangle = -\int d^n x J_\mu^R \frac{\delta Z^R}{\delta J_\mu^R} + 2\langle \int d^n x O_7^R \rangle + \langle \int d^n x O_5^R \rangle$$

$$\langle \int d^n x O_5^R \rangle = \int d^n x \; \overline{\eta}_a(x) \frac{\delta Z^R}{\delta \overline{\eta}^a(x)} = \int d^n x O_5$$

$$\langle \int d^n x O_6^R \rangle = \frac{1}{2} e \frac{\partial Z^R}{\partial e} + \frac{1}{2} \int d^n x \; J_\mu^R \frac{\delta Z^R}{\delta J_\mu^R}$$

$$\langle \int d^n x O_7^R \rangle = -\xi \frac{\partial Z^R}{\partial \xi} \tag{6.9}$$

The definition of O_1^R holds only upto $O(e^2)$, because λ is not

multiplicatively renormalizable(This has been explained in detail in the context of scalar Q.E.D.). However, Eq.(6.9) suffices for our purpose since our treatment is in $O(e^2)$ only.

Thus, the trace equation can be written as

$$\langle\theta_\mu{}^{imp\mu}\rangle^{U.R.} = \text{finite} - \varepsilon \sum_{j=1}^{8} Z_{1j}\langle O_j\rangle^R + \tilde{g}\, Z_m^{-1}\langle\partial^2(\phi^T\phi)\rangle^R \tag{6.10}$$

Z_{1j}, $j=1,2,\cdots,7$ can be obtained by considering the above equation at zero momentum:

$$\langle\int d^n x\theta_\mu{}^{imp\mu}\rangle^{U.R.} = \text{finite} - \varepsilon\sum_{j=1}^{8} Z_{1j}\langle\int d^n x O_j\rangle^R \tag{6.11}$$

Following exactly the same steps as in Chapter 5, one obtains

$$\begin{aligned}
Z_{11} &= 1 - \frac{\beta^\lambda}{\lambda\varepsilon} & Z_{14} &= \frac{\gamma_3 - \beta^e}{\varepsilon} \\
Z_{12} &= \frac{\gamma_m}{\varepsilon} & Z_{15} &= \frac{\beta^e}{e\varepsilon} - \frac{\gamma_3}{\varepsilon} - 2\frac{\tilde{\gamma}}{\varepsilon} \\
Z_{13} &= \frac{\gamma}{\varepsilon} & Z_{16} &= \frac{\beta^\lambda}{\lambda\varepsilon} - \frac{2\beta^e}{e\varepsilon} \\
Z_{17} &= \frac{\gamma_3}{\varepsilon}
\end{aligned} \tag{6.12}$$

where the RG functions are defined as follows:

$$\begin{aligned}
\beta^\lambda(\lambda,e,\varepsilon) &= -\lambda\varepsilon + \beta^\lambda \equiv \mu\frac{\partial\lambda}{\partial\mu}\Big|_{bare} \\
\beta^e(\lambda,e,\varepsilon) &= -\frac{e\varepsilon}{2} + \beta^e \equiv \mu\frac{\partial e}{\partial\mu}\Big|_{bare} \\
\gamma_m(\lambda,e,\varepsilon) &= \gamma_m(\lambda,e) \equiv \frac{1}{2}\mu\frac{\partial}{\partial\mu}\ln Z_m\Big|_{bare} \\
\gamma(\lambda,e,\xi,\varepsilon) &= \gamma(\lambda,e,\xi) \equiv \frac{1}{2}\mu\frac{\partial}{\partial\mu}\ln Z\Big|_{bare} \\
\gamma_3(\lambda,e,\xi,\varepsilon) &= \gamma_3(\lambda,e,\xi) \equiv \frac{1}{2}\mu\frac{\partial}{\partial\mu}\ln Z_3\Big|_{bare} \\
\tilde{\gamma}(\lambda,e,\xi,\varepsilon) &= \tilde{\gamma}(\lambda,e,\xi) \equiv \frac{1}{2}\mu\frac{\partial}{\partial\mu}\ln\tilde{Z}\Big|_{bare}
\end{aligned} \tag{6.13}$$

[In contrast to the previous case, we have taken here the two

mass parameters to be equal, $\mu_1=\mu_2=\mu$] Thus, one obtains the trace equation at zero momentum,

$$\langle\int d^n x\ \theta_\mu^{imp\mu}\rangle^{U.R.} = \frac{\beta^\lambda(\lambda,e)}{\lambda}\langle\int d^n x\ O_1\rangle^R - \gamma_m(\lambda,e)\langle\int d^n x\ O_2\rangle^R$$
$$- \gamma(\lambda,e)\langle\int d^n x O_3\rangle^R + (\beta^e - \gamma_3)\langle\int d^n x O_4\rangle^R$$
$$-(\beta^e - 2\tilde{\gamma} - \gamma_3)\langle\int d^n x O_5\rangle^R + (2\beta^e - \beta^\lambda)\langle\int d^n x O_6\rangle^R$$
$$- \gamma_3\langle\int d^n x\ O_7\rangle^R \tag{6.14}$$

Thus, $\theta_\mu^{imp\mu}$ is finite at zero momentum. As Z_{1j}'s, $j=1,2,\cdots\cdots,7$ have only simple poles at zero momentum, the trace equation at non-zero momentum reduces to

$$\langle\theta_\mu^{imp\mu}\rangle = \text{finite} + [-\varepsilon Z_{18} + \tilde{g}Z_m^{-1}]\langle\partial^2(\phi^T\phi)\rangle^R \tag{6.15}$$

Z_{18} is as yet unknown, but we have verified by explicit calculations that εZ_{18} is finite only upto $O(\lambda^3)$ at $e=0$, upto $O(e^4)$ at $\lambda=0$ and also in $O(\lambda e^2)$. However, in $O(\lambda^4)$, $O(\lambda e^4)$ and $O(\lambda^2 e^2)$ Z_{18} does have double poles[2] and therefore a non- zero $\tilde{g}$ is necessarily needed to obtain a finite $\theta_{\mu\nu}^{imp}$. We will now consider the possibility of choosing a $\tilde{g}$ such that the quantity

$$X = Z_{18} + gZ_m^{-1} \tag{6.16}$$

where $g = -\frac{1}{\varepsilon}\tilde{g}$, has no worse than simple poles, this being the necessary condition for finiteness of $\theta_{\mu\nu}^{imp}$. We will discuss both forms [A] and [B] (discussed earlier) for g and will show, in each case, that it is impossible to make $\theta_{\mu\nu}^{imp}$ finite.

CASE I : Consider $\theta_{\mu\nu}^{imp}$ of the following form

$$\theta_{\mu\nu}^{imp} = \theta_{\mu\nu} + \left[\frac{n-2}{4(1-n)} + \frac{\tilde{g}(\varepsilon, e_o^2\mu^{-\varepsilon}, \lambda_o\mu^{-\varepsilon})}{(1-n)}\right](\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2)(\phi^T\phi) \tag{6.17}$$

where

$$\tilde{g}(\varepsilon, e_o^2\mu^{-\varepsilon}, \lambda_o\mu^{-\varepsilon}) \equiv -\ \varepsilon g(\varepsilon, e_o^2\mu^{-\varepsilon}, \lambda_o\mu^{-\varepsilon})$$

$$= \sum_{n=0}^{\infty} g_n(\varepsilon,\lambda_o\mu^{-\varepsilon})(e_o^2\mu^{-\varepsilon})^n \qquad (6.18)$$

εg_n's are finite functions of their arguments, but g_n's can have $1/\varepsilon$ terms.

To prove the negative result [A], we shall use the RG equation satisfied by Z_{18}. Z_{ij} satisfies the following RG equation,

$$Z^{-1}_{ij}\ \mu\frac{\partial}{\partial\mu}Z_{jk} = \gamma_{ik} = \text{finite at } \varepsilon=0 \qquad (6.19)$$

Using the fact that

(i) O_2, O_3 and O_5 are finite operators.

(ii) O_4 can mix under renormalization only with O_3, O_4 and O_5 (This has been shown in Appendix H).

(iii) O_7 can mix only with O_3, O_4 and O_5 (also shown in Appendix H).

(iv) $\partial^2(\phi^T\phi)$ is a multiplicatively renormalizable operator:

$$\langle\partial(\phi^T\phi)\rangle^{U.R.} = Z_m^{-1}\langle\partial^2(\phi^T\phi)\rangle^{R}$$

one obtains the following structure for $\mathbb{Z}$:

$$\mathbb{Z} = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} & Z_{15} & Z_{16} & Z_{17} & Z_{18} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & Z_{43} & Z_{44} & Z_{45} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ Z_{61} & Z_{62} & Z_{63} & Z_{64} & Z_{65} & Z_{66} & Z_{67} & Z_{68} \\ 0 & 0 & Z_{73} & Z_{74} & Z_{75} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & Z_m^{-1} \end{bmatrix} \qquad (6.20)$$

Eqs.(6.19) and (6.20) yield, following the same procedure as in Appendix G , the following RG equation to be satisfied by Z_{18}:

$$(-\lambda\varepsilon + \beta^{\lambda})\frac{\partial Z_{18}}{\partial\lambda} + (-\frac{e\varepsilon}{2} + \beta^{e})\frac{\partial Z_{18}}{\partial e} - 2\gamma_m Z_{18} = Z_{11}\gamma_{18} + Z_{16}\gamma_{68}$$

(6.21)

Eq.(6.21), when combined with the RG equation for Z_m^{-1}[see Eq.(4.16)], implies

$$(-\lambda\varepsilon + \beta^{\lambda})\frac{\partial X}{\partial\lambda} + (-\frac{e\varepsilon}{2} + \beta^{e})\frac{\partial X}{\partial e} - 2\gamma_m X - Z_{11}\gamma_{18} - Z_{1\sigma}\gamma_{\sigma 8}$$
$$= -\varepsilon\sum_{n=0}^{\infty}(e_o^2\mu^{-\varepsilon})^n\left\{ng_n + \frac{\partial g_n}{\partial(\lambda_o\mu^{-\varepsilon})}\lambda_o\mu^{-\varepsilon}\right\} Z_m^{-1} \qquad (6.22)$$

Now, suppose it were possible to choose g_n's such that X has no worse than simple poles(which, in turn, would imply the existence of a finite energy momentum tensor). Then, as Z_{11} and $Z_{1\sigma}$ have only simple poles, the left hand side of Eq.(6.22) has at worst simple poles and hence, so has the right hand side. Hence,

$$\varepsilon^2\left[\sum_{n=0}^{\infty}(e_o^2\mu^{-\varepsilon})^n\left\{ng_n + \frac{\partial g_n}{\partial(\lambda_o\mu^{-\varepsilon})}\lambda_o\mu^{-\varepsilon}\right\} Z_m^{-1}\right] = \text{finite} \qquad (6.23)$$

which is the same as Eq.(5.49) of Chapter 5. It has already been shown in Chapter 5 that the above equation when considered in $O(e^2)$ implies that

$$g_1(\varepsilon,\lambda_o\mu^{-\varepsilon}) = 0 \qquad (6.24)$$

Therefore, the improvement term under consideration is consistent with the finiteness of $\langle\theta_{\mu}^{\,\iota mp\mu}\rangle$ in $O(e^2\lambda^n)$ only if

$$g(\varepsilon,e_o^2\mu^{-\varepsilon},\lambda_o\mu^{-\varepsilon}) = g_o(\varepsilon) + O(e^4) \qquad (6.25)$$

i.e. the improvement term obtained to $O(e^0)$ is sufficient even to $O(e^2)$. But we have verified by explicit calculations[2] that an additional term is necessarily needed in $O(e^2)$ to make $\theta_{\mu\nu}$ finite. Hence, we conclude that it is not possible to find an improved energy momentum tensor of the form in Eq.(6.17) which may be finite even to $O(e^2\lambda^n)$.

CASE II: Consider $\theta^{imp}_{\mu\nu}$ of the following form

$$\theta^{imp}_{\mu\nu} = \theta_{\mu\nu} + \left[\frac{n-2}{4(1-n)} + \frac{\tilde{g}(\varepsilon,e^2,\lambda)}{(1-n)}\right](\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2)(\phi^T\phi) \qquad (6.26)$$

where

$$\tilde{g}(\varepsilon,e^2,\lambda) = -\varepsilon g(\varepsilon,e^2,\lambda) = -\varepsilon\sum_{n=0}^{\infty} e^{2n} g_n(\lambda,\varepsilon) \qquad (6.27)$$

$\varepsilon g_n(\varepsilon,\lambda)$'s being finite functions of λ, although they may have $1/\varepsilon$ terms. Using Eq.(6.21) and

$$\mu\frac{\partial}{\partial\mu} g(\varepsilon,e^2,\lambda) = (-\lambda\varepsilon + \beta^\lambda)\frac{\partial g}{\partial\lambda} + (-\frac{e\varepsilon}{2} + \beta^e)\frac{\partial g}{\partial e}$$

one obtains the following equation satisfied by X,

$$(-\lambda\varepsilon + \beta^\lambda)\frac{\partial X}{\partial\lambda} + (-\frac{e\varepsilon}{2} + \beta^e)\frac{\partial X}{\partial e} - 2\gamma_m X - Z_{11}\gamma_{18} - Z_{16}\gamma_{68}$$
$$= \left[(-\lambda\varepsilon + \beta^\lambda)\frac{\partial g}{\partial\lambda} + (-\frac{e\varepsilon}{2} + \beta^e)\frac{\partial g}{\partial e}\right] Z_m^{-1} \qquad (6.28)$$

As before, the existence of finite $\theta^{imp\mu}_{\mu}$ implies that

$$\varepsilon\left[(-\lambda\varepsilon + \beta^\lambda)\frac{\partial g}{\partial\lambda} + (-\frac{e\varepsilon}{2} + \beta^e)\frac{\partial g}{\partial e}\right] Z_m^{-1} = \text{finite} \qquad (6.29)$$

Eq.(6.29) is exactly similar to Eq.(5.63) and hence, when considered in $O(e^2)$ implies that

$$g(\varepsilon,e^2,\lambda) = g_0(\varepsilon) + O(e^4) \qquad (6.30)$$

which, as discussed in Chapter 5, leads to an inconsistency. Hence, it is impossible to find an improved energy momentum tensor of the form given in Eq.(6.26) also.

[6.3]ENERGY MOMENTUM TENSOR IN YUKAWA THEORY

In this section, we shall consider the possibility of a finite improvement program in the context of Yukawa theory of scalar-fermion interaction. The Lagrangian density is given

by,[1]

$$L = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}M_o^2\phi^2 - \frac{\lambda_o\phi^4}{4!} + \bar\psi(\iota\not\partial - m_o)\psi + \iota g_o\bar\psi\gamma_5\psi\phi \qquad (6.27)$$

The symmetric energy momentum tensor, given by,

$$\theta_{\mu\nu} = -g_{\mu\nu}L + \partial_\mu\phi\partial_\nu\phi + \frac{\iota}{4}\left[\bar\psi\gamma_\mu(\overrightarrow{\partial}_\nu - \overleftarrow{\partial}_\nu)\psi + \bar\psi\gamma_\nu(\overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu)\psi\right] \qquad (6.28)$$

has finite matrix elements at q=0 and to first order in q, but in $O(q^2)$ a further improvement is needed[3].The most general improvement one can add to $\theta_{\mu\nu}$ is parametrized as

$$\theta^{imp}_{\mu\nu} = \theta_{\mu\nu} + \left[\frac{n-2}{4(1-n)} + \frac{\tilde G}{1-n}\right](\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2)\phi^2 \qquad (6.29)$$

As in the case of scalar ϕ^4– theory, finiteness of $\theta^{imp\mu}_{\mu}$ is sufficient to prove the finiteness of $\theta^{imp}_{\mu\nu}$. Trace of $\theta^{imp}_{\mu\nu}$ is given by

$$\theta^{imp}_{\mu\nu} = (n-4)\left[-\frac{\lambda_o\phi^4}{4!} + \frac{\iota}{2}g_o\psi\bar\gamma_5\psi\phi\right] + (M_o^2\phi^2 + m_o\bar\psi\psi) - \left(\frac{n-2}{2}\right)\phi\frac{\delta S}{\delta\phi}$$

$$-(n-1)\bar\psi\frac{\delta S}{\delta\bar\psi} + \tilde G\partial^2\phi^2 \qquad (6.30)$$

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[1]The renormalization transformations are

$$\phi = Z^{1/2}\phi^R \qquad\qquad \psi = \tilde Z^{1/2}\psi^R$$

$$\lambda_o = \mu^\varepsilon[\lambda Z_\lambda + \delta\lambda(g)] \qquad\qquad \bar\psi = \tilde Z^{1/2}\bar\psi^R$$

$$g_o = \mu^{\varepsilon/2}gZ_g \qquad\qquad m_o = Z_m m$$

$$M_o^2 = Z_M M^2 + Z_M{}' m^2 \qquad (F.1)$$

λ is not multipicatively renormalizable for the same reason as in scalar Q.E.D. case. $\delta\lambda$ starts with $O(g^4)$. Here, the scalar mass is also not multiplicatively renormalizable due to the presence of diagrams for the scalar propagator(such as the single fermion loop diagram in $O(g^2)$) which give contributions proportional to m^2. $Z_M{}'$ starts with $O(g^2)$.

As shown in Chapter 4, $M_o^2\phi^2$ and $\phi\frac{\delta S}{\delta\phi}$ are finite operators and $\partial^2\phi^2$ has the renormalization,[2]

$$\left\{\partial^2\phi^2\right\}^{U.R} = Z_m^{-1}\left\{\partial^2\phi^2\right\}^R \tag{6.31}$$

One can show along the same lines as Eqs.(4.13) and (5.6) that $m_o\bar{\psi}\psi$ and $\bar{\psi}\frac{\delta S}{\delta\bar{\psi}}$ are also finite operators. Thus,

$$\langle\theta_\mu^{imp\mu}\rangle^{U.R.} = \text{finite} + (n-4)\langle O_1\rangle^{U.R.} + \tilde{G}Z_m^{-1}\langle\partial^2\phi^2\rangle^R \tag{6.32}$$

where

$$O_1 = -\frac{\lambda_o\phi^4}{4!} + \frac{i}{2}\,g_o\bar{\psi}\gamma_5\psi\phi .$$

Thus, we need to know the renormalization of O_1. O_1 can mix under renormalization with the following set of operators:

$$O_2 = M_o^2\phi^2$$
$$O_3 = m_o\bar{\psi}\psi$$
$$O_4 = \phi\,\frac{\delta S}{\delta\phi}$$
$$O_5 = \bar{\psi}\,\frac{\delta S}{\delta\bar{\psi}}$$
$$O_6 = \frac{1}{2}\,i g_o\bar{\psi}\gamma_5\psi\phi \tag{6.33}$$
$$O_7 = \partial^2\phi^2$$

Renormalized operators are defined as follows:

$$\langle\int d^n x\ O_1\rangle^R = \lambda\frac{\partial Z^R}{\partial\lambda} + \frac{1}{2}\,g\,\frac{\partial Z^R}{\partial g}$$

---

[2] ϕ^2 is multiplicatively renormalizable operator, since there is no dimension two operator it can mix with. Its renormalization constant can be obtained by dividing the equation

$$\langle M_o^2\phi^2\rangle = -2M_o^2\frac{\partial Z^{U.R}}{\partial M_o^2} = -2\left(1+\frac{m^2}{M^2}\,\frac{Z_m}{Z_M}\right)M^2\frac{\partial Z^R}{\partial M^2} \tag{F.2}$$

by M^2 and putting m=0, as the renormalization constant is mass independent in M-S scheme.

$$\langle\int d^n x O_2\rangle^R = -2M^2 \frac{\partial Z^R}{\partial M^2}$$

$$\langle\int d^n x O_3\rangle^R = -m \frac{\partial Z^R}{\partial m}$$

$$\langle \int d^n x O_4\rangle^R = -\int d^n x J^R \frac{\delta Z^R}{\delta J^R}$$

$$\langle\int d^n x O_5\rangle^R = -\int d^n x\, \bar{\eta}^R(x) \frac{\delta Z^R}{\delta \bar{\eta}^R(x)}$$

$$\langle\int d^n x O_6\rangle^R = -\int d^n x\, \eta^R(x) \frac{\delta Z^R}{\delta \eta^R(x)}$$

$$\langle\int d^n x O_7\rangle^R = \frac{1}{2} g \frac{\partial Z^R}{\partial g} \tag{6.34}$$

where $\eta(x)$ and $\bar{\eta}(x)$ are the sources for the fermion fields. Again these definitions hold only upto $O(e^2)$, but that suffices for our purpose. The renormalization matrix is defined by

$$\langle O_i\rangle^{U.R.} = \sum_{j=1}^{8} Z_{ij} \langle O_j\rangle^R \tag{6.35}$$

Thus, we have

$$\langle \theta_\mu{}^{\iota mp\mu}\rangle^{U.R.} = \text{finite} - \varepsilon \sum_{j=1}^{8} Z_{1j} \langle O_j\rangle^R + \tilde{G}\, Z_m^{-1} \langle \partial^2 \phi^2\rangle^R \tag{6.36}$$

Z_{1j}, j=1,2,……,7 can be obtained by considering $\langle\int d^n x\ O_1^R\rangle$ and retracing the same steps as Eqs.(5.28) -(5.33) in Chapter 5. The final result is

$$Z_{11} = 1 - \frac{\beta^\lambda}{\lambda\varepsilon} \qquad Z_{14} = \frac{\gamma}{\varepsilon}$$

$$Z_{12} = \frac{\gamma_M}{\varepsilon} \qquad Z_{15} + Z_{16} = 2\frac{\tilde{\gamma}}{\varepsilon}$$

$$Z_{13} = \frac{\gamma_m}{\varepsilon} \qquad Z_{17} = \frac{\beta^\lambda}{\lambda\varepsilon} - \frac{2\beta^g}{g\varepsilon} \tag{6.37}$$

One may note that the above procedure does not yield Z_{15} and Z_{16} separately because $\int d^n x\ O_5 = \int d^n x O_6$. From Eq.(6.37), it only follows that $Z_{15} + Z_{16}$ has simple poles. However, the

theory has a charge conjugation invariance and the operator $O_5 - O_6 = i\partial^\mu(\bar\psi\gamma_\mu\psi)$, which is odd under charge conjugation, cannot appear as a counterterm for O_1. Hence, only the combination $O_5 + O_6$ can appear in the expression for $O_1^{U.R.}$. This requires that

$$Z_{15} = Z_{16} = \frac{\tilde\gamma}{\varepsilon}$$

i.e. both Z_{15} and Z_{16} have only simple poles in ε. Thus, we obtain the trace equation at zero momentum:

$$\begin{aligned}\langle \int d^n x\, \theta_\mu^{\;imp\mu}\rangle^{U.R.} = {} & \frac{\beta^\lambda(\lambda,g)}{\lambda}\langle\int d^n x\; O_1\rangle^R - \gamma_M(\lambda,g)\langle\int d^n x\; O_2\rangle^R \\ & - \gamma_m(\lambda,g)\langle\int d^n x O_3\rangle^R - \gamma(\lambda,g)\langle\int d^n x O_4\rangle^R \\ & - \tilde\gamma(\lambda,g)\langle\int d^n x O_5 + \int d^n x O_6\rangle^R \\ & -\left(\frac{\beta^\lambda}{\lambda} - \frac{2\beta^g}{g}\right)\langle\int d^n x\; O_7\rangle^R \qquad (6.38)\end{aligned}$$

where the RG functions are defined as below:

$$\begin{aligned}\beta^\lambda(\lambda,g,\varepsilon) &= -\lambda\varepsilon + \beta^\lambda \equiv \mu\frac{\partial\lambda}{\partial\mu}\Big|_{bare} \\ &= -\lambda\varepsilon + \lambda^2\frac{\partial Z_\lambda^{(1)}}{\partial\lambda} + \lambda\frac{\partial(\delta\lambda^{(1)})}{\partial\lambda} + \frac{g}{2}\frac{\partial(\delta\lambda^{(1)})}{\partial g} - \delta\lambda^{(1)} \\ &\quad + \frac{1}{2}\lambda g\frac{\partial Z_\lambda^{(1)}}{\partial g} \\ &= -\lambda\varepsilon + \beta_2\lambda^2 + \cdots\cdots \\ \beta^g(\lambda,g,\varepsilon) &= -\frac{g\varepsilon}{2} + \beta^g \equiv \mu\frac{\partial g}{\partial\mu}\Big|_{bare} \\ &= -\frac{\varepsilon g}{2} + \lambda g\frac{\partial Z_g^{(1)}}{\partial\lambda} + \frac{g^2}{2}\frac{\partial Z_g^{(1)}}{\partial g} \\ \gamma_M(\lambda,g,\varepsilon) &= \gamma_M(\lambda,g) \equiv \frac{1}{2}\mu\frac{\partial}{\partial\mu}\ln M^2\Big|_{bare} \\ \gamma_m(\lambda,g,\varepsilon) &= \gamma_m(\lambda,g) \equiv \frac{1}{2}\mu\frac{\partial}{\partial\mu}\ln m^2\Big|_{bare} \\ \tilde\gamma_M(\lambda,g,\varepsilon) &= \tilde\gamma_M(\lambda,g) \equiv -\frac{1}{2}\mu\frac{\partial}{\partial\mu}\ln Z_M\Big|_{bare} \\ &= \frac{\lambda}{2}\frac{\partial Z_M^{(1)}}{\partial\lambda} + \frac{g}{4}\frac{\partial Z_M^{(1)}}{\partial g} = \gamma_M^{(1)}\lambda + \cdots\cdots\end{aligned}$$

$$\gamma(\lambda,g,\varepsilon) = \gamma(\lambda,g) \equiv \frac{1}{2}\mu\frac{\partial}{\partial\mu}\ln Z\Big|_{bare}$$

$$\tilde{\gamma}(\lambda,g,\varepsilon) = \tilde{\gamma}(\lambda,g) \equiv \frac{1}{2}\mu\frac{\partial}{\partial\mu}\ln\tilde{Z}\Big|_{bare} \tag{6.39}$$

Thus, $\theta^{\iota mp\mu}_{\mu}$ is finite at zero momentum. One can verify by explicit calculations that at non-zero momentum, it is finite only upto $O(\lambda^3)$ at $g=0$, upto $O(g^4)$ at $\lambda=0$ and also in $O(\lambda g^2)$, but an improvement term is necessarily needed in $O(\lambda^4)$, $O(\lambda g^4)$ and $O(\lambda^2 g^2)$[3]. We shall now consider improvement term dependence of both types [A] and [B] stated in Sec.[6.1] and will check if either of these forms can lead to a finite energy momentum tensor.

CASE I: Consider an improved energy momentum tensor of the form

$$\theta^{\iota mp}_{\mu\nu} = \theta_{\mu\nu} + \left[\frac{n-2}{4(1-n)} + \frac{\tilde{G}(\varepsilon,g_0^2\mu^{-\varepsilon},\lambda_0\mu^{-\varepsilon})}{(1-n)}\right](\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2)\phi^2 \tag{6.40}$$

where

$$\tilde{G}(\varepsilon,g_0^2\mu^{-\varepsilon},\lambda_0\mu^{-\varepsilon}) \equiv -\varepsilon G(\varepsilon,g_0^2\mu^{-\varepsilon},\lambda_0\mu^{-\varepsilon})$$
$$= -\varepsilon\sum_{n=0}^{\infty} G_n(\varepsilon,\lambda_0\mu^{-\varepsilon})(g_0^2\mu^{-\varepsilon})^n$$

$\tilde{G}$ is a finite function of its arguments, whereas G and G_n can have $1/\varepsilon$ terms also. Using Eqs.(6.36) and (6.37), one obtains the following trace equation,

$$\langle\theta^{\iota mp\mu}_{\mu}\rangle = \text{finite} + [-\varepsilon Z_{18} + \tilde{G}Z_m^{-1}]\langle\partial^2\phi^2\rangle^R \tag{6.41}$$

Defining

$$X = Z_{18} + G(\varepsilon,g_0^2\mu^{-\varepsilon},\lambda_0\mu^{-\varepsilon})Z_m^{-1}$$

one obtains

$$\langle\theta^{\iota mp\mu}_{\mu}\rangle^{U.R} = \text{finite} - \varepsilon X\langle\partial^2\phi^2\rangle^R \tag{6.42}$$

Renormalization matrix $\mathbb{Z}$ has the following structure[3]:

$$\mathbb{Z} = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} & Z_{15} & Z_{16} & Z_{17} & Z_{18} \\ 0 & Z_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & Z_{32} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ Z_{71} & Z_{72} & Z_{73} & Z_{74} & Z_{75} & Z_{76} & Z_{77} & Z_{78} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & Z_m^{-1} \end{bmatrix}$$

which, together with,

$$Z^{-1}_{ij}\ \mu\frac{\partial}{\partial\mu}Z_{jk} = \gamma_{ik} = \text{finite at } \varepsilon=0$$

gives the RG equation satisfied by Z_{18}[4],

$$(-\lambda\varepsilon + \beta^{\lambda})\frac{\partial Z_{18}}{\partial\lambda} + (-\frac{g\varepsilon}{2} + \beta^{g})\frac{\partial Z_{18}}{\partial g} - 2\tilde{\gamma}_M Z_{18} = Z_{11}\gamma_{18} + Z_{17}\gamma_{78} \tag{6.43}$$

Hence, as in Sec[6.2], X satisfies the following RG equation,

$$(-\lambda\varepsilon + \beta^{\lambda})\frac{\partial X}{\partial\lambda} + (-\frac{g\varepsilon}{2} + \beta^{g})\frac{\partial X}{\partial g} - 2\tilde{\gamma}_M X - Z_{11}\gamma_{18} - Z_{17}\gamma_{78}$$
$$= -\varepsilon\sum_{n=0}^{\infty}(g_0^2\mu^{-\varepsilon})^n\left\{nG_n + \frac{\partial G_n}{\partial(\lambda_0\mu^{-\varepsilon})}\lambda_0\mu^{-\varepsilon}\right\} Z_M^{-1} \tag{6.44}$$

[3]Here, we have used Eq.(6.31) and the fact that

(i) $\phi\frac{\delta S}{\delta\phi}$, $\bar{\psi}\frac{\delta S}{\delta\bar{\psi}}$ and $\frac{\delta S}{\delta\phi}\phi$ are finite operators.

(ii) Eq.(F.2) implies that $Z_{22} \neq 1$.

(iii) $m_0\bar{\psi}\psi$ mixes with $M_0^2\phi^2$:

$$\langle m_0\bar{\psi}\psi\rangle = -m_0\frac{\partial Z^{U.R.}}{\partial m_0} = -m\frac{\partial Z^R}{\partial m} + (\frac{2m^2}{M^2} + \frac{Z'_M}{Z_M})M^2\frac{\partial Z^R}{\partial M^2} \tag{F.3}$$

[4]Here, γ_m and $\tilde{\gamma}_m$ are not the same in contrast to the scalar Q.E.D. and NAGT's cases. The reason lies in the fact that M_0^2 is not multiplicatively renormalizable. It is easy to verify that

$$\tilde{\gamma}_M = \gamma_M + \frac{m^2}{M^2}\frac{Z'_M}{Z_M}(\gamma_m + \frac{1}{2}\mu\frac{\partial}{\partial\mu}\ln Z'_m) \tag{F.4}$$

Now, if it were possible to choose G_n's such that X has no worse than simple poles then, the above equation implies that

$$\varepsilon^2\left[\sum_{n=0}^{\infty}(g_0^2\mu^{-\varepsilon})^n\left\{nG_n + \frac{\partial G_n}{\partial(\lambda_0\mu^{-\varepsilon})}\lambda_0\mu^{-\varepsilon}\right\} Z_M^{-1}\right] = \text{finite}$$

which being the same as Eq.(5.49) implies that

$$G(\varepsilon,\lambda_0\mu^{-\varepsilon},g_0^2\mu^{-\varepsilon}) = G_0(\varepsilon) + O(g^4)$$

meaning thereby that no further improvement is needed in $O(g^2)$. This contradicts the result obtained by direct calculation[3]. Hence, an improvement of type [A] cannot yield a finite energy momentum tensor.

CASE II: Consider, now the $\theta_{\mu\nu}^{imp}$ of the following form

$$\theta_{\mu\nu}^{imp} = \theta_{\mu\nu}+\left[\frac{n-2}{4(1-n)} + \frac{\tilde{G}(\varepsilon,g^2,\lambda)}{(1-n)}\right](\partial_\mu\partial_\nu-g_{\mu\nu}\partial^2)\phi^2 \tag{6.45}$$

where

$$\tilde{G}(\varepsilon,g^2,\lambda) = -\varepsilon G(\varepsilon,g^2,\lambda) = -\varepsilon\sum_{n=0}^{\infty} g^{2n}G_n(\lambda,\varepsilon)$$

$G_n(\varepsilon,\lambda)$'s can have $1/\varepsilon$ terms also. Then

$$\langle\theta_\mu^{imp\mu}\rangle^{U.R.} = \text{finite} - \varepsilon X\langle\partial^2\phi^2\rangle^R \tag{6.46}$$

where X is given by

$$X = Z_{18}+ G(\varepsilon,e^2,\lambda)Z_M^{-1} \tag{6.47}$$

X satisfies the following equation,

$$\begin{aligned}(-\lambda\varepsilon + \beta^\lambda)\frac{\partial X}{\partial\lambda} +(-\frac{g\varepsilon}{2} + \beta^g)\frac{\partial X}{\partial g} - 2\tilde{\gamma}_M X- Z_{11}\gamma_{18}- Z_{17}\gamma_{78}\\ = \left[(-\lambda\varepsilon + \beta^\lambda)\frac{\partial G}{\partial\lambda} +(-\frac{g\varepsilon}{2} + \beta^g)\frac{\partial G}{\partial g}\right]Z_M^{-1}\end{aligned} \tag{6.48}$$

As before, the existence of finite $\theta_\mu^{imp\mu}$ implies that

$$\varepsilon\left[(-\lambda\varepsilon + \beta^\lambda)\frac{\partial G}{\partial\lambda} +(-\frac{g\varepsilon}{2} + \beta^g)\frac{\partial G}{\partial g}\right]Z_M^{-1} = \text{finite} \tag{6.49}$$

Eq.(6.49) is exactly similar to Eq.(5.63) and hence, when considered in $O(g^2)$ implies that

$$G(\varepsilon, e^2, \lambda) = G_0(\varepsilon) + O(g^4) \tag{6.50}$$

which, as discussed in Chapter 5, leads to an inconsistency. Hence, it is impossible to find an improved energy momentum tensor of the form [B] also which may be finite to $O(g^2\lambda^n)$.

[6.4]ENERGY MOMENTUM TENSOR IN A MODEL WITH TWO SCALARS

In this section, we shall establish the impossibility of having a finite improvement program in a theory involving two interacting scalar fields. We shall restrict ourselves to the case of only two independent couplings and we shall assume the masses of the two scalar fields to be the same: this simplifies the treatment without altering the conclusion. We consider the Lagrangian density,

$$L = \frac{1}{2}(\partial_\mu \phi_1)^2 - \frac{1}{2}m_0^2\phi_1^2 - \frac{\lambda_0\phi_1^4}{4!} + \frac{1}{2}(\partial_\mu \phi_2)^2 - \frac{1}{2}m_0^2\phi_2^2 - \frac{\lambda_0\phi_2^4}{4!} - \frac{1}{4}\varkappa_0\phi_1^2\phi_2^2 \tag{6.51}$$

The canonical energy momentum tensor

$$\theta^C_{\mu\nu} = -g_{\mu\nu}L + \partial_\mu\phi_1\partial_\nu\phi_1 + \partial_\mu\phi_2\partial_\nu\phi_2$$

does not have finite matrix elements beyond $O(\lambda)$. We define an improved energy momentum tensor

$$\theta^{imp}_{\mu\nu} = \theta^C_{\mu\nu} + \left[\frac{n-2}{4(1-n)} + \frac{\tilde{g}}{1-n}\right](\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2)(\phi_1^2 + \phi_2^2) \tag{6.52}$$

As in the case of scalar ϕ^4- theory, finiteness of $\theta^{imp\mu}_\mu$ is sufficient to prove the finiteness of $\theta^{imp}_{\mu\nu}$. Trace of $\theta^{imp}_{\mu\nu}$ is given by

$$\langle\theta^{imp\mu}_\mu\rangle^{U.R.} = (n-4)\left\langle\left[-\frac{\lambda_0\phi_1^4}{4!} - \frac{\lambda_0\phi_2^4}{4!} - \frac{1}{4}\varkappa_0\phi_1^2\phi_2^2\right]\right\rangle^{U.R.}$$

$$+\langle m_o^2\phi_1^2 + m_o^2\phi_2^2\rangle^{U.R} - \left(\frac{n-2}{2}\right)\langle\phi_1 \frac{\delta S}{\delta\phi_1} + \phi_2 \frac{\delta S}{\delta\phi_2}\rangle^{U.R.}$$

$$+ \tilde{g}\langle\partial^2(\phi_1^2 + \phi_2^2)\rangle^{U.R.} \tag{6.53}$$

To obtain $\langle\theta_\mu^{\ \ imp\mu}\rangle^{U.R.}$ one needs the renormalization of operators appearing in the above equation. As shown in Chapter 4 for one scalar, $m_o^2(\phi_1^2 + \phi_2^2)$ and $\phi_1\frac{\delta S}{\delta\phi_1} + \phi_2\frac{\delta S}{\delta\phi_2}$ here are also finite operators and $\partial^2(\phi_1^2 + \phi_2^2)$ has the renormalization

$$\left\{\partial^2(\phi_1^2 + \phi_2^2)\right\}^{U.R} = Z_m^{-1}\left\{\partial^2(\phi_1^2 + \phi_2^2)\right\}^R$$

$\left(-\frac{\lambda_o\phi^4}{4!} - \frac{\lambda_o\phi^4}{4!} - \frac{1}{4}\varkappa_o\phi_1^2\phi_2^2\right)$ mixes under renormalization with the following operators:

$$O_1 = -\frac{\lambda_o\phi_1^4}{4!} - \frac{\lambda_o\phi_2^4}{4!} - \frac{1}{4}\varkappa_o\phi_1^2\phi_2^2$$

$$O_2 = m_o^{\ 2}(\phi_1^2 + \phi_2^2)$$

$$O_3 = \phi_1\frac{\delta S}{\delta\phi_1}$$

$$O_4 = \phi_2\frac{\delta S}{\delta\phi_2}$$

$$O_5 = -\frac{1}{4}\varkappa_o\phi_1^2\phi_2^2$$

$$O_6 = \partial^2(\phi_1^2 + \phi_2^2) \tag{6.54}$$

The above set is closed under renormalization. Renormalized operators are defined as follows:

$$\langle\int d^n x\ O_1\rangle^R = \lambda\frac{\partial Z^R}{\partial\lambda} + \varkappa\frac{\partial Z^R}{\partial\varkappa}$$

$$\langle\int d^n x O_2\rangle^R = -2m^2\frac{\partial Z^R}{\partial m^2}$$

$$\langle\int d^n x O_3\rangle^R = -\int d^n x\ J_1^R\frac{\delta Z^R}{\delta J_1^R}$$

$$\langle \int d^n x O_4 \rangle^R = -\int d^n x \; J_2^R \frac{\delta Z^R}{\delta J_2^R}$$

$$\langle \int d^n x O_5 \rangle^R = \varkappa \frac{\partial Z^R}{\partial \varkappa} \tag{6.55}$$

[The definition of O_1^R is valid only to $O(\varkappa)$. This is due to the fact that λ is not multiplicatively renormalizable. However, our treatment needs only $O(\varkappa)$ quantities and hence, the above definition is correct for our purpose].

The renormalization matrix is defined by

$$\langle O_i \rangle^{U.R} = Z_{ij} \langle O_j \rangle^R$$

where

$$\mathbf{Z} = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} & Z_{15} & Z_{16} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ Z_{51} & Z_{52} & Z_{53} & Z_{54} & Z_{55} & Z_{56} \\ 0 & 0 & 0 & 0 & 0 & Z_m^{-1} \end{bmatrix} \tag{6.56}$$

Thus, we have

$$\langle \theta_\mu^{\;imp\mu} \rangle^{U.R.} = \text{finite} - \varepsilon \sum_{j=1}^{6} Z_{1j} \langle O_j \rangle^R + \tilde{g} Z_m^{-1} \langle \partial^2 \phi^2 \rangle^R \tag{6.57}$$

Z_{1j}, $j=1,2,\cdots\cdots,5$ can be obtained by considering $\langle \int d^n x \; O_1^R \rangle$ and they are found to have only simple poles:

$$Z_{11} = 1 - \frac{\beta^\lambda}{\lambda\varepsilon} \qquad Z_{13} = \frac{\gamma}{\varepsilon} = Z_{14}$$

$$Z_{12} = \frac{\gamma_m}{\varepsilon} \qquad Z_{15} = \frac{\beta^\lambda}{\lambda\varepsilon} - \frac{\beta^\varkappa}{\varkappa\varepsilon} \tag{6.58}$$

Hence, at zero momentum $\langle \theta_\mu^{\;imp\mu} \rangle$ is finite. At non zero momentum, Eq.(6.57) reduces to

$$\langle \theta_\mu^{\;imp\mu} \rangle^{U.R} = \text{finite} - \varepsilon X \langle \partial^2 \phi^2 \rangle^R \tag{6.59}$$

where

$$X = Z_{16} - \frac{\tilde{g}}{\varepsilon} Z_m^{-1} = Z_{16} + g Z_m^{-1}$$

To obtain a finite $\theta_{\mu\nu}^{imp}$, one should be able to choose a g such that X has no worse than simple poles. We will now show that it is not possible to do so in a consistent manner with either an improvement term of type [A] or an improvement term of type [B].

CASE I: Consider an improved energy momentum tensor of the form

$$\theta_{\mu\nu}^{imp} = \theta_{\mu\nu} + \left[\frac{n-2}{4(1-n)} + \frac{\tilde{g}(\varepsilon, \varkappa_o \mu^{-\varepsilon}, \lambda_o \mu^{-\varepsilon})}{(1-n)}\right](\partial_\mu \partial_\nu - g_{\mu\nu}\partial^2)(\phi_1^2 + \phi_2^2) \tag{6.60}$$

where

$$\tilde{g}(\varepsilon, \varkappa_o \mu^{-\varepsilon}, \lambda_o \mu^{-\varepsilon}) \equiv -\varepsilon g(\varepsilon, \varkappa_o \mu^{-\varepsilon}, \lambda_o \mu^{-\varepsilon})$$
$$= -\varepsilon \sum_{n=0}^{\infty} g_n(\varepsilon, \lambda_o \mu^{-\varepsilon})(\varkappa_o \mu^{-\varepsilon})^n \tag{6.61}$$

$\tilde{g}$ is a finite function of its arguments, whereas g and g_n can have $1/\varepsilon$ terms also. From Eq.(6.56) and the RG equation for $\mathbf{Z}$,

$$Z_{ij}^{-1} \mu \frac{\partial}{\partial \mu} Z_{jk} = \gamma_{ik} = \text{finite at } \varepsilon = 0 \tag{6.62}$$

one easily obtains the RG equation satisfied by Z_{16}[4],

$$(-\lambda\varepsilon + \beta^\lambda)\frac{\partial Z_{16}}{\partial \lambda} + (-\varkappa\varepsilon + \beta^\varkappa)\frac{\partial Z_{16}}{\partial \varkappa} - 2\gamma_m Z_{16} = Z_{11}\gamma_{16} + Z_{15}\gamma_{56} \tag{6.63}$$

where the relevant RG functions are defined below:

$$\beta^\lambda(\lambda, \varkappa, \varepsilon) = -\lambda\varepsilon + \beta^\lambda \equiv \mu \frac{\partial \lambda}{\partial \mu}\Big|_{bare}$$

$$\beta^\varkappa(\lambda, \varkappa, \varepsilon) = -\varkappa\varepsilon + \beta^\varkappa \equiv \mu \frac{\partial \varkappa}{\partial \mu}\Big|_{bare}$$

$$\gamma_m(\lambda, \varkappa, \varepsilon) = \gamma_m(\lambda, \varkappa) \equiv -\frac{1}{2}\mu \frac{\partial}{\partial \mu} \ln Z_m\Big|_{bare} \tag{6.64}$$

Eqs.(6.63) and (4.16) when combined together yield the RG

equation satisfied by X

$$(-\lambda\varepsilon + \beta^{\lambda})\frac{\partial X}{\partial\lambda} + (-\varkappa\varepsilon + \beta^{\varkappa})\frac{\partial X}{\partial\varkappa} - 2\gamma_m X - Z_{11}\gamma_{1\sigma} - Z_{15}\gamma_{5\sigma}$$

$$= -\varepsilon\sum_{n=0}^{\infty}(\varkappa_o\mu^{-\varepsilon})^n\left\{ng_n(\varepsilon,\lambda_o\mu^{-\varepsilon}) + \frac{\partial g_n(\varepsilon,\lambda_o\mu^{-\varepsilon})}{\partial(\lambda_o\mu^{-\varepsilon})}\lambda_o\mu^{-\varepsilon}\right\} Z_m^{-1} \tag{6.65}$$

As discussed in earlier sections, finiteness of $\theta^{imp\mu}_{\mu}$ requires that the right hand side of Eq.(6.65) have no worse than simple poles, which in turn, implies that

$$g(\varepsilon,\lambda_o\mu^{-\varepsilon},\varkappa_o\mu^{-\varepsilon}) = g_o(\varepsilon) + O(\varkappa^2) \tag{6.66}$$

which contradicts the result obtained by direct calculation[4]. Hence, an improvement of type [A] cannot yield a finite energy momentum tensor.

CASE II: Consider, now $\theta^{imp}_{\mu\nu}$ of the following form

$$\theta^{imp}_{\mu\nu} = \theta_{\mu\nu} + \left[\frac{n-2}{4(1-n)} + \frac{\tilde{g}(\varepsilon,\varkappa,\lambda)}{(1-n)}\right](\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2)(\phi_1^2 + \phi_2^2) \tag{6.67}$$

where

$$\tilde{g}(\varepsilon,\varkappa,\lambda) = -\varepsilon g(\varepsilon,\varkappa,\lambda) = -\varepsilon\sum_{n=0}^{\infty}\varkappa^n g_n(\lambda,\varepsilon)$$

$g_n(\varepsilon,\lambda)$'s can have $1/\varepsilon$ terms also. Then X satisfies the following equation,

$$(-\lambda\varepsilon + \beta^{\lambda})\frac{\partial X}{\partial\lambda} + (-\varkappa\varepsilon + \beta^{\varkappa})\frac{\partial X}{\partial\varkappa} - 2\gamma_m X - Z_{11}\gamma_{1\sigma} - Z_{15}\gamma_{5\sigma}$$

$$= \left[(-\lambda\varepsilon + \beta^{\lambda})\frac{\partial g}{\partial\lambda} + (-\varkappa\varepsilon + \beta^{\varkappa})\frac{\partial g}{\partial\varkappa}\right] Z_m^{-1} \tag{6.68}$$

X will have no worse than simple poles provided the right hand side of Eq.(6.68) has no worse than simple poles. This implies, when leading dependence of β^{λ} and $\beta^{\varkappa}$ is taken into account, that

$$g(\varepsilon,\varkappa,\lambda) = g_o(\varepsilon) + O(\varkappa^2)$$

which leads to a contradiction.Hence, it is impossible to find an improved energy momentum tensor of the form [B] also which may be finite to $O(\varkappa\lambda^n)$.

[6.5]CONCLUSION

It has been shown in Chapters 5 and 6 that,in all the four theories under consideration viz.scalar Q.E.D., NAGT's with scalars, Yukawa theory and a theory of two interacting scalar fields, it is impossible to have a finite improvement program. This means that one has to necessarily add extra infinite counterterms to make the matrix elements of the energy momentum tensor finite and hence, in these theories, one needs to determine an extra parameter from the experiment to define the theory completely.

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APPENDIX A

In this appendix, we shall prove the identity of Eq. (2.45) viz.for $\lambda_m \neq 0$, and any operator X,

$$\phi_m^{\dagger}(x)X\gamma_5\phi_m(x) = -\frac{1}{2\lambda_m}\left\{\phi_m^{\dagger}(x)[X,\not{D}]\gamma_5\phi_m(x)+\partial^{\mu}[\phi_m^{\dagger}(x)\gamma_{\mu}X\gamma_5\phi_m(x)]\right\} \tag{2.45}$$

$$\begin{aligned}\phi_m^{\dagger}(x)X\gamma_5\phi_m(x) &= \frac{1}{\lambda_m}\ \phi_m^{\dagger}(x)X\gamma_5\not{D}\phi_m(x) \\ &=-\frac{1}{\lambda_m}\ \phi_m^{\dagger}(x)X\not{D}\gamma_5\phi_m(x) \\ &=-\frac{1}{\lambda_m}\left\{\ \phi_m^{\dagger}(x)[X,\not{D}]\gamma_5\phi_m(x)+\phi_m^{\dagger}(x)\not{D}X\gamma_5\phi_m(x)]\right\}\end{aligned} \tag{A.1}$$

Now using

$$\phi_m^{\dagger}(x)[-\overleftarrow{\partial} + A] = \lambda_m\phi_m^{\dagger}(x) \tag{A.2}$$

one obtains for any Y,

$$\phi_m^{\dagger}\not{D}Y = \lambda_m\phi_m^{\dagger}Y + \partial^{\mu}(\phi_m^{\dagger}\ \gamma_{\mu}Y) \tag{A.3}$$

Using (A.3) in (A.1) and simplifying, one obtains Eq.(2.45).

APPENDIX B

In this appendix, we shall derive the general form of the operator O(2n) stated in Sec.[2.3]

From Eq.(2.42), we have

$$O(2n) = \lim \frac{1}{(n-2)!} \frac{\partial^{n-2}}{\partial (M^{-2})^{n-2}} A_M(x) \qquad n \geq 3$$

$$= \lim \frac{1}{(n-2)!} \sum_m (-\lambda_m^2)^{n-2} \phi_m^\dagger(x) \exp(-\lambda_m^2/M^2) \gamma_5 \phi_m(x) \qquad (B.1)$$

We shall first prove by induction that $(-\lambda_m^2)^{n-2}\, \phi_m^\dagger(x)\gamma_5\phi_m(x)$ has the form $(n \geq 3; \lambda_m \neq 0)$

$$(-\lambda_m^2)^{n-2}\phi_m^\dagger(x)\gamma_5\phi_m(x) = \partial^\mu\partial^\nu\left\{\phi_m^\dagger K^{(n)}_{\mu\nu}\gamma_5\phi_m\right\} + $$
$$+\left(-\frac{1}{4}\right)^{n-2}\partial^\mu\left\{\phi_m^\dagger(x)[[\cdots[\gamma_\mu \not{D}],\not{D}]\cdots\not{D}]\gamma_5\phi_m(x)\right\}$$
$$(B.2)$$

where the last term has (2n-5) commutators. For n = 3, the result is true by virtue of Eq.(2.46).

Let the result be true for n = r. In other words, let

$$(-\lambda_m^2)^{r-2}\phi_m^\dagger(x)\gamma_5\phi_m(x) = \partial^\mu\ \partial^\nu\left\{\phi_m^\dagger K^{(r)}_{\mu\nu}\gamma_5\phi_m\right\} + $$
$$+ \left(-\frac{1}{4}\right)^{r-2}\ \partial^\mu\left\{\phi_m^\dagger(x)\underbrace{[[\cdots[\gamma_\mu,\not{D}],\not{D}]\cdots\not{D}]}_{2r-5\ \text{commutators}}\gamma_5\phi_m(x)\right\}$$
$$(B.3)$$

for some $K_r^{\mu\nu}$. We shall prove the result for n = r+1. Consider

$$(-\lambda_m^2)^{r-1}\phi_m^\dagger(x)\gamma_5\phi_m(x) = (-\lambda_m^2)\left\{(-\lambda_m^2)^{r-2}\phi_m^\dagger\gamma_5\phi_m\right\}$$
$$= -\lambda_m^2\left\{\partial^\mu\ \partial^\nu\left\{\phi_m^\dagger K^{(r)}_{\mu\nu}\gamma_5\phi_m\right\} + \right.$$
$$\left. +\left(-\frac{1}{4}\right)^{r-2}\partial^\mu\left\{\phi_m^\dagger(x)[[\cdots[\gamma_\mu \not{D}],\not{D}]\cdots\not{D}]\gamma_5\phi_m(x)\right\}\right.$$

(B.4)

Applying Eq.(2.45) to the second term on the right hand side,

$$\partial^{\mu}\Big\{\phi_m^{\dagger}(x)[[\cdots[\gamma_{\mu},\not{D}],\not{D}]\cdots\not{D}]\gamma_5\phi_m(x)\Big\}$$

$\longleftarrow 2r-5 \text{ — commutators} \longrightarrow$

$$=\partial^{\mu}\Big\{-\frac{1}{2\lambda_m}\phi_m^{\dagger}(x)[[\cdots[\gamma_{\mu},\not{D}],\not{D}]\cdots\not{D}]\gamma_5\phi_m(x)$$

$\longleftarrow 2r-4 \text{ — commutators} \longrightarrow$

$$-\frac{1}{2\lambda_m}\partial^{\nu}[\phi_m^{\dagger}(x)\gamma_{\nu}[[\cdots[\gamma_{\mu},\not{D}],\not{D}]\cdots\not{D}]\gamma_5\phi_m(x)]\Big\} \qquad (B.5)$$

$\longleftarrow 2r \quad -5 \text{ — commutators} \longrightarrow$

Applying Eq.(2.40) once more to the first term on the right hand side one obtains

$$-\lambda_m^2\partial^{\mu}(\phi_m^{\dagger}(x)[[\cdots[\gamma_{\mu},\not{D}],\not{D}]\cdots\not{D}]\gamma_5\phi_m(x)\Big\}$$

$$= -\frac{1}{4}\partial^{\mu}\Big\{\phi_m^{\dagger}(x)[[\cdots[\gamma_{\mu},\not{D}],\not{D}]\cdots\not{D}]\gamma_5\phi_m(x)\Big\}$$

$\longleftarrow 2r-3 \text{ — commutators} \longrightarrow$

$$-\frac{1}{4}\partial^{\mu}\partial^{\nu}\Big\{\phi_m^{\dagger}(x)\gamma_{\nu}[[\cdots[\gamma_{\mu},\not{D}],\not{D}]\cdots\not{D}]\gamma_5\phi_m(x)\Big\} \qquad (B.6)$$

$\longleftarrow 2r-4 \text{ — commutators} \longrightarrow$

while the second term in Eq.(B.5) yields

$$(-\lambda_m^2)\Big(-\frac{1}{2\lambda_m}\Big)\partial^{\mu}\partial^{\nu}\Big\{\phi_m^{\dagger}(x)\gamma_{\nu}[[\cdots[\gamma_{\mu},\not{D}],\not{D}]\cdots\not{D}]\gamma_5\phi_m(x)\Big\}$$

$$= \frac{1}{2}\partial^{\mu}\partial^{\nu}\Big\{\phi_m^{\dagger}(x)\gamma_{\nu}[[\cdots[\gamma_{\mu},\not{D}],\not{D}]\cdots\not{D}]\gamma_5\not{D}\,\phi_m(x)\Big\} \qquad (B.7)$$

$\longleftarrow 2r \text{ — } 5 \text{ — commutators} \longrightarrow$

Combining Eqs.(B.5),(B.6),(B.7) in Eq.(B.4) one obtains

$$(-\lambda_m^2)^{r-1}\phi_m^{\dagger}\gamma_5\phi_m$$

$$=\partial^{\mu}\partial^{\nu}\Big\{\phi_m^{\dagger}\;K_{\mu\nu}^{(r)}\gamma_5\not{D}^2\phi_m+\Big(-\frac{1}{4}\Big)^{r-1}\phi_m^{\dagger}\gamma_{\nu}[[\cdots[\gamma_{\mu},\not{D}],\not{D}]\cdots\not{D}]\gamma_5\phi_m$$

$\longleftarrow 2r-4 \text{ — commutators} \longrightarrow$

$$+\frac{1}{2}\Big(-\frac{1}{4}\Big)^{r-3}\phi_m^{\dagger}\gamma_{\nu}[[\cdots[\gamma_{\mu},\not{D}],\not{D}]\cdots\not{D}]\gamma_5\not{D}\phi_m\Big\}$$

$\longleftarrow 2r-5 \text{ — commutators} \longrightarrow$

$$+\Big(-\frac{1}{4}\Big)^{r-2}\partial^{\mu}\Big\{\phi_m^{\dagger}(x)[[\cdots[\gamma_{\mu},\not{D}],\not{D}]\cdots\not{D}]\gamma_5\phi_m(x)\Big\}$$

$\longleftarrow 2r-3 \text{ — commutators} \longrightarrow$

$$\equiv\partial_{\mu}\partial_{\nu}\Big\{\phi_m^{\dagger}K_{\mu\nu}^{(r+1)}\gamma_5\phi_m$$
$$+(-\tfrac{1}{4})^{r-1}\partial^{\mu}\Big\{\phi_m^{\dagger}(x)[[\cdots[\gamma_{\mu},\not{D}],\not{D}]\cdots\not{D}]\gamma_5\,\phi_m(x)\Big\}$$

where

$$K_{\mu\nu}^{(r+1)}=K_{\mu\nu}^{(r)}\not{D}^2\ +(-\tfrac{1}{4})^{r-1}\gamma_{\nu}[[\cdots[\gamma_{\mu},\not{D}],\not{D}]\cdots\not{D}]$$
$$\leftarrow 2r-4\text{—commutators}\rightarrow$$
$$-\tfrac{1}{2}(-\tfrac{1}{4})^{r-2}\gamma_{\nu}[[\cdots[\gamma_{\mu},\not{D}],\not{D}]\cdots\not{D}]\not{D}$$

with $\quad K_{\mu\nu}^{(3)}=-\frac{1}{4}\ g_{\mu\nu}1$ (B.9)

In Eq.(B.8) we have proved the result of Eq.(B.2) for n = r+1.Hence the proof of Eq.(B.2) by induction is complete.

Using the result of Eq.(B.2) in Eq.(B.1), we obtain

$$O(2n)\ =\ \lim_{M^2\to\infty}\Big\{\partial_{\mu}\partial_{\nu}\sum_{\lambda_m\neq 0}\ \phi_m^{\dagger}K_{\mu\nu}^{(n)}\gamma_5\exp(-\not{D}^2/M^2)\phi_m$$
$$+(-\tfrac{1}{4})^{n-2}\partial^{\mu}\Big\{\sum_{\lambda_m\neq 0}\phi_m^{\dagger}(x)[[\cdots[\gamma_{\mu},\not{D}],\not{D}]\cdots\not{D}]\gamma_5\exp(-\not{D}^2/M^2)\phi_m(x)\Big\}$$

(B.10)

The restriction $\lambda_m\neq 0$ can be removed as in the case of O(6).(See Eqs.(2.48)).To see this, we consider the additional terms needed to remove the restriction in the curly bracket of Eq. (B.10). They are

$$\partial_{\mu}\partial_{\nu}\sum_{\lambda_m=0}\phi_m^{\dagger}K_{\mu\nu}^{(n)}\gamma_5\phi_m+(-\tfrac{1}{4})^{n-2}\partial^{\mu}\Big\{\sum_{\lambda_m=0}\phi_m^{\dagger}(x)[[\cdots[\gamma_{\mu},\not{D}],\not{D}]\cdots\not{D}]\gamma_5\phi_m(x)$$

These can be shown to vanish as in the case of O(6) using Eq. (B.9). We thus have

$$O(2n)\ =\ \partial_{\mu}\partial_{\nu}O_{\mu\nu}^{(1)}(2n-2)\ +\ \partial^{\mu}O_{\mu}^{(2)}(2n-1) \qquad \text{(B.11)}$$

where

$$O_{\mu\nu}^{(1)}(2n-2)\ =\ \lim_{M^2\to\infty}\frac{1}{(n-2)!}\sum_m\phi_m^{\dagger}K_{\mu\nu}^{(n)}\gamma_5\exp(-\not{D}^2/M^2)\phi_m$$

and

$$O^{(2)}_{\mu}(2n-1) = \lim_{M^2\to\infty} \frac{1}{(n-2)!}\left(-\frac{1}{4}\right)^{n-2}\Big\{\sum_{\lambda_m} \phi^{\dagger}_m(x)\underbrace{[[\cdots[\gamma_\mu, \not{D}], \not{D}]\cdots \cdots \not{D}]}_{\leftarrow 2n-5 \text{ commutators} \rightarrow} \times\exp(-\not{D}^2/M^2)\gamma_5\phi_m(x)\Big\} \qquad (B.13)$$

To prove the result stated at the beginning of Sec.[2.3], we have to show that $O^{(1)}$ and $O^{(2)}$ are local, gauge invariant operators of gauge fields. To this end, we first note that on account of Eq.(B.9), $K^{(n)}_{\mu\nu}$ is a polynomial in $\not{D}$ say $K^{(n)}_{\mu\nu}(\not{D})$. Thus $O^{(1)}_{\mu\nu}(2n-2)$ can be evaluated as done for the anomaly viz;

$$O^{(1)}_{\mu}(2n-2) = \lim_{M^2\to\infty} \frac{1}{(n-2)!} \sum_m \phi^{\dagger}_m(x)K^{(n)}_{\mu\nu}(\not{D})\gamma_5 \exp(-\not{D}^2/M^2)\phi_m(x)$$

$$= \lim_{M^2\to\infty} \frac{1}{(n-2)!}\int\frac{d^n k}{(2\pi)^4} K^{(n)}_{\mu\nu}(\not{D} + i\not{k})\exp[-(\not{D} + i\not{k})^2/M^2]$$

(B.14)

In the limit $M^2 \to \infty$, only a finite number of terms contribute to $O^{(1)}_{\mu\nu}(2n-2)$and the resultant expression is local in A_μ.A similar argument applies to $O^{(2)}_{\mu}(2n-1)$.

To prove the gauge invariance of $O^{(1)}_{\mu\nu}(2n-2)$and $O^{(2)}_{\mu}(2n-1)$, we note that they are both of the form

$$\sum \phi^{\dagger}_m(x)f(\not{D})\exp(-\lambda^2_m/M^2)\phi_m(x)$$

where $f(\not{D})$ is a matrix polynomial in $\not{D}$.

This is invariant under a gauge transformation as under a gauge transformation, the basis functions ϕ_m change to

$$\phi_m(x) \to \phi_m{}'(x) = \exp[i\alpha(x)]\phi_m(x);$$

$$\phi^{\dagger}_m(x) \to \phi^{\dagger}_m{}'(x) = \phi^{\dagger}_m{}'(x)\exp[-i\alpha(x)]$$

whereas

$$f(\not{D})\phi_m(x) \to f(\not{D}')\phi_m{}'(x) = \exp[i\alpha(x)]f(\not{D})\phi_m(x) \qquad (B.15)$$

Hence $O^{(1)}_{\mu\nu}(2n-2)$ and $O^{(2)}_{\mu}(2n-1)$ are gauge invariant.

APPENDIX C

In this appendix, we show that $f_n(g^2, \ln M^2)$ occuring in Eq.(2.54) are non-trivial (i.e.individually non-zero) by working out f_3 to the lowest non-trivial order in g^2 viz.$O(g^4)$.

From Eq.(2.52), $\tilde{O}^{(2)}_{\mu}(5)$ has the form

$$\tilde{O}^{(2)}_{\mu}(5) = Cg_o^2 D^{ab\eta} F^b_{\eta\delta} \tilde{F}^{a\delta}_{\mu} = C' g_o^2 \frac{\delta S_o}{\delta A^a_\delta} \tilde{F}^{a\delta}_{\mu} \tag{C.1}$$

where C and C' are constants and $S_o = \frac{1}{2g^2} \int d^4x \, Tr(F_{\mu\nu}F^{\mu\nu})$.

We can express $\tilde{O}^{(2)}(5)$ as

$$\tilde{O}^{(2)}_{\mu}(5) = C' g_o^2 \left[\frac{\delta S_{eff}}{\delta A^a_\delta} \tilde{F}^{a\delta}_{\mu} - i g_o \bar{\psi} \gamma^\delta T^a \psi \, F^{a\delta}_{\mu} - \frac{\delta S_{ghost}}{\delta A^a_\delta} \tilde{F}^{a\delta}_{\mu} \right] \tag{C.2}$$

We shall show that $\partial^\mu \langle O^{(2)}_{\mu}(5)\rangle$ leads to a non-trivial divergence when one considers the two-fermion matrix elements of $\partial^\mu O^{(2)}_{\mu}(5)$ in one loop approximation. This would mean a non-trivial $f_3(g^2, \ln M^2)$ to $O(g^4)$.

Consider the two-fermion matrix element of $\tilde{O}^{(2)}_{\mu}(5)$ of Eq.(C.2) in the one loop approximation.The last term involving S_{ghost} does not contribute to one loop diagrams as fermions do not couple to ghosts directly. On account of equations of

motion valid to one loop approximation we have

$$\left\langle \frac{\delta S_{eff}}{\delta A^a_\delta} \tilde{F}^{a\mu\delta} \right\rangle = -J^a_\delta \langle \tilde{F}^{a\mu\delta} \rangle = \frac{\delta\Gamma}{\delta \langle A^a_\delta \rangle^R} \langle \hat{F}^{a\mu\delta} \rangle$$

$$= \frac{\delta\Gamma}{\delta \langle A^a_\delta \rangle^R} \langle Z^{-1/2} \tilde{F}^{a\mu\delta} \rangle. \qquad (C.3)$$

Thus, the effective dimensions of this term in $O^{(2)}_\mu(5)$ is reduced from five to two as $\delta\Gamma/\delta\langle A^a_\delta\rangle^R$ is finite. This term has at best $\ln M^2$ divergences coming from $Z^{-1/2}$. Thus, it does not contribute to $\lim_{M^2\to\infty} \frac{\langle O(6)\rangle}{M^2}$.

It is straightforward to verify that the middle term in the right hand side of Eq.(C.2) does indeed lead to quadratically divergent two-fermion proper vertex. This leads to a divergence of $O(g_0^4 M^2)$. Thus $f_s(g^2, \ln M^2) \neq 0$ to $O(g^4)$.

APPENDIX D

Here we shall show that $\Delta^{(0)}$ of Eq.(4.47) is non-zero by diagonalization of Δ which turns out to be straightforward enough. We shall give a sequence of operations which will diagonalize $\Delta^{(0)}$.

By operation $R_a - xR_b$ we shall mean subtracting from a^{th} row the x^{th} multiple of the b^{th} row. This operation, of course, leaves a determinant unchanged.

Now we perform successively $R_{p+1} - \frac{1}{2p+1} R_p$,...., $R_r - \frac{1}{p+r} R_{r-1}$, ..., $R_2 - \frac{1}{p+2} R_1$. We take a factor of $\frac{1}{p+r}$ common from the r^{th} row ($r = 2, 3, \ldots, p+1$). The result is

$$\Delta^{(0)} \longrightarrow \Delta^{(1)} = \left(\prod_{r=2}^{p+1} \frac{1}{p+r}\right)\Delta^{(1)} \qquad \text{(D.1)}$$

where

$$\Delta^{(1)} = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & \frac{1}{p+1} & \frac{2}{p} & \cdots & \frac{p}{2} \\ 0 & \frac{1}{(p+1)(p+2)} & \frac{2}{p(p+1)} & \cdots & \frac{p}{2.3} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & \frac{1}{(p+1)\ldots(2p)} & \frac{2}{p\ldots(2p-1)} & \cdots & \frac{p}{(p+1)!} \end{vmatrix}$$

(D.2)

We perform similar operations on the pxp submatrix. We shall

proceed by induction. Let after ι operations we have the determinant $\Delta^{(\iota)}$ of the form

$$\Delta^{(\iota)} = \left| \begin{array}{ccccccc} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 0 & \frac{1}{p+1} & \frac{2}{p} & \cdots & \frac{\iota-1}{p-\iota+3} & \frac{\iota}{p-\iota+2} & \frac{p}{2} \\ 0 & 0 & \frac{2}{p(p+1)} & \cdots & \frac{(\iota-1)(\iota-2)}{(p+3-\iota)(p+4-\iota)} & \cdots & \frac{(p-1)(p-2)}{p(p+1)} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \frac{(\iota-1)!}{(p+3-\iota)\dots(p+1)} & \cdots & \cdot \\ 0 & 0 & 0 & & 0 & & \\ \vdots & \vdots & \vdots & & \vdots & & \\ 0 & 0 & 0 & & 0 & & \end{array} \right|$$

$\longleftarrow$ l columns $\longrightarrow$

(D.3)

where the $(p+1-\iota)x((p+1-\iota)$ principal minor at the bottom is given in particular by

$$\Delta^{(\iota)}_{rk} = \frac{(k-1)\dots\dots\dots(k-\iota)}{(p+r+1-k)\dots\dots(p+3-k)}, \quad \begin{array}{l} r = (\iota+1),\dots\dots,(p+1) \\ s = 1,\ 2,\ \dots\dots,(p+1) \end{array}$$

(D.4)

Then the $(\iota+1)^{th}$ operation

$$R_{p+1} - \frac{1}{2p-\iota+1} R_p \ ; \quad \dots\dots \quad R_r - \frac{1}{p-\iota+r} R_{r-1}\ ; \quad \dots\dots$$
$$R_{\iota+2} - \frac{1}{p+2} R_{\iota+1}$$

leads after taking a factor of $\frac{1}{p-\iota+r}$ common from r^{th} row to $\Delta^{(\iota+1)}$ where

$$\Delta^{(\iota)} \longrightarrow \Delta^{(\iota+1)} = \prod_{r=\iota+2}^{p+1} \frac{1}{p-\iota+r}\ \Delta^{(\iota+r)} \qquad (D.5)$$

and $\Delta^{\iota+1}$ has the same form as $\Delta^{(\iota)}$ but with $\iota \longrightarrow \iota+1$ and the $(p-\iota)x(p-\iota)$ principal minor at the bottom given by

$$\Delta^{(\iota+1)}_{rk} = \frac{(k-1)\dots\dots\dots(k-\iota-1)}{(p+r+1-k)\dots\dots(p+3-k)}\ ; \quad \begin{array}{l} r = (\iota+2)\dots\dots\dots(p+1) \\ k = \iota,\ 2,\dots\dots\dots(p+1) \end{array}$$

(D.6)

The above staement has been verified for $\iota=0$ above. To prove it for arbitrary ι, we start from $\Delta^{(\iota)}_{rk}$ of Eq.(D.4) and perform the operations given below it. Then $\Delta^{(\iota)} \longrightarrow \Delta^{(\iota+1)'}$ where

$$\Delta^{(\iota+1)'}_{rk} = \Delta^{(\iota)}_{rk} - \frac{1}{p-\iota+r} \Delta^{(\iota)}_{r-1,k} \qquad \left\{ \begin{array}{l} r = (\iota+2), \ \ldots(p+1) \\ s = 1, \ 2, \ \ldots.(p+1) \end{array} \right\}$$

$$= \frac{(k-1)(k-2)\ldots\ldots\ldots\ldots(k-\iota-1)}{(p+r-k)\ldots\ldots(p+3-k)(p+r+1-k)(p-\iota+r)}$$

Taking factor of $\frac{1}{(p-\iota+r)}$ common from r^{th} row $[(\iota+2)\le r\le p+1]$ we see that the result in (D.6) above is proved.

As a result of p such successive operations one ends up with $\Delta^{(p)}$ whose diagonal elements are all non zero [See Eq.(D.4) for $\Delta^{(\iota)}_{\iota+1,\iota+1}$]. Hence $\Delta^{(p)}$ and $\Delta^{(0)}$ are non zero. This proves the result.

APPENDIX E

In this appendix, we will show explicitely (for the case of scalar Q.E.D.) that X does have double poles to $O(\lambda^2 e^2)$, thus proving that for no choice of $\tilde{g}$ of either form chosen in sections [5.5] and [5.6] does X have no poles higher than simple poles.

It is shown by Brown that $g_0(\varepsilon)$ begins with $O(\varepsilon^2)$. Therefore, $g_0(\varepsilon)Z_m^{-1}$ does not have worse than simple poles upto $O(\lambda^2 e^2)$. Hence, double poles in X to $O(\lambda^2 e^2)$ arise entirely from Z_{17}.

One may verify by direct calculation that Z_{17} has no worse than simple poles in $O(\lambda e^2)$ and that

$$Z^{(1)}_{17(1,1)} = \frac{1}{16\pi^2}\left(-\frac{3}{2}\right) \tag{E.1}$$

where $Z^{(r)}_{17(m,n)}$ denotes the coefficient of $\dfrac{\lambda e^{2n}}{\varepsilon^r}$ in Z_{17}.

The RGE for Z_{17} can be written using Eq.(5.35) as

$$\left(-\lambda\varepsilon + \beta^{\lambda}_1 + \beta^{\lambda}_2\right)\frac{\partial}{\partial\lambda}Z_{17} + \left(-\frac{e\varepsilon}{2} + \beta^{e}_1 + \beta^{e}_2\right)\frac{\partial}{\partial e}Z_{17} - 2(\gamma_{m_1} + \gamma_{m_2})Z_{17}$$
$$= \left[1 - \frac{\beta^{\lambda}_1 + \beta^{\lambda}_2}{\lambda\varepsilon}\right](\gamma_{17} + \gamma'_{17}) + \left[\frac{\beta^{\lambda}_1 + \beta^{\lambda}_2}{\lambda\varepsilon} - 2\frac{\beta^{e}_1 + \beta^{e}_2}{e\varepsilon}\right](\gamma_{57} + \gamma'_{57}) \tag{E.2}$$

But

$$\gamma_{17} + \gamma'_{17} = -\lambda\frac{\partial}{\partial\lambda}Z^{(1)}_{17} - \frac{e}{2}\frac{\partial}{\partial e}Z^{(1)}_{17}$$
$$\gamma_{57} + \gamma'_{57} = -\lambda\frac{\partial}{\partial\lambda}Z^{(1)}_{57} - \frac{e}{2}\frac{\partial}{\partial e}Z^{(1)}_{57} \tag{E.3}$$

as seen easily from Eq.(E.2) and the corrresponding RG equation for Z_{57} respectively. Therefore Eq.(E.2) can be rewritten as

$$(-\lambda\varepsilon + \beta_1^\lambda + \beta_2^\lambda)\frac{\partial}{\partial\lambda}Z_{17} + (-\frac{e\varepsilon}{2} + \beta_1^e + \beta_2^e)\frac{\partial}{\partial e}Z_{17} - 2(\gamma_{m_1} + \gamma_{m_2})Z_{17}$$

$$= -\lambda\frac{\partial}{\partial\lambda}Z_{17}^{(1)} - \frac{e}{2}\frac{\partial}{\partial e}Z_{17}^{(1)}$$

$$+\left[\frac{\beta_1^\lambda+\beta_2^\lambda}{\lambda\varepsilon}\right]\left[\lambda\frac{\partial}{\partial\lambda}(Z_{17}^{(1)} - Z_{57}^{(1)}) + \frac{e}{2}\frac{\partial}{\partial e}(Z_{17}^{(1)} - Z_{57}^{(1)})\right]$$

$$+2\left[\frac{\beta_1^e+\beta_2^e}{e\varepsilon}\right]\lambda\frac{\partial}{\partial\lambda}Z_{57}^{(1)} + \left[\frac{\beta_1^e+\beta_2^e}{\varepsilon}\right]e\frac{\partial}{\partial e}Z_{57}^{(1)} \tag{E.4}$$

which in $O(e^2)$ reduces to

$$(-\lambda\varepsilon + \beta_1^\lambda + \beta_2^\lambda)\frac{\partial}{\partial\lambda}Z_{17} - \frac{e\varepsilon}{2}\frac{\partial}{\partial e}Z_{17} - 2(\gamma_{m_1} + \gamma_{m_2})Z_{17}$$

$$= \text{finite} + \left[\frac{\beta_1^\lambda+\beta_2^\lambda}{\varepsilon}\right]\left[\frac{\partial}{\partial\lambda}(Z_{17}^{(1)} - Z_{57}^{(1)})\right]$$

$$+ 2\left[\frac{\beta_1^e+\beta_2^e}{e\varepsilon}\right]\lambda\frac{\partial}{\partial\lambda}Z_{57}^{(1)} + \frac{\beta_1^\lambda}{\lambda\varepsilon}\frac{e}{2}\frac{\partial}{\partial e}(Z_{17}^{(1)} - Z_{57}^{(1)}) + O(e^3)$$

$$\tag{E.5}$$

comparing the coeffcients of $\frac{\lambda^2 e^2}{\varepsilon}$ on both sides of Eq.(E.5) one obtains,

$$-3\, Z_{17(2,1)}^{(2)} = (\beta_2 - 2\gamma_{m_1}^{(1)})Z_{17(1,1)}^{(1)} \tag{E.6}$$

where we have used that

(i) $(Z_{17}-Z_{57})$ has no poles in $O(\lambda^0 e^2)$ because $O_1 - O_5 = -\frac{\lambda_0(\phi^*\phi)}{4}$ is proportional t0 λ upto $O(e^2)$.

(iii) At $e=0$, Z_{17} begins as λ^3.

(iv) $(Z_{17} - Z_{57})$ has no poles in $O(\lambda e^2)$, because $\frac{\lambda_o(\phi^*\phi)^2}{4!}$ does not need counter terms of the form $\partial^2(\phi^*\phi)$ in $O(\lambda e^2)$.

But [See Eq. (5.19)]

$$(\beta_2 - 2\gamma^{(1)}_{m_1}) \neq 0$$

therefore Eqs. (E.1) and (E.6) together imply that Z_{17} (and hence $X = Z_{17} + g_o(\varepsilon) Z_m^{-1}$) <u>does</u> have double poles in $O(\lambda^2 e^2)$. Therefore, the improvement coefficient $g_o(\varepsilon)$ does not work in $O(\lambda^2 e^2)$.

APPENDIX F

In this appendix, we will show(in the context of scalar Q.E.D.) that the operator $\frac{1}{2}\xi_o(\partial.A)^2$ mixes under renormalization only with operators which vanish by classical equations of motion viz. O_3 and O_4 of chapter 5 to all orders. We shall also show that the operator $\xi_o\partial^\rho(A_\rho\partial.A)$ is a finite operator to all orders.

We perform the following gauge transformation on the integration variables of the generating functional of Eq.(5.2):

$$
\begin{aligned}
A_\mu(x) &\longrightarrow A_\mu(x) + \partial_\mu\Lambda(x) \\
\phi(x) &\longrightarrow [1 + \iota e\Lambda(x)]\ \phi(x) \\
\phi^*(x) &\longrightarrow [1 - \iota e\Lambda(x)]\ \phi^*(x)
\end{aligned}
\tag{F.1}
$$

where we let $\Lambda(x) = \int G(x,y)\theta[A(y)]d^ny$. Here, θ is an infinitesimal quantity and may depend on gauge fields. $G(x,y)$ is defined by

$$(\partial_x^2-\iota\varepsilon)\ G(x,y) = \delta^n(x-y) \tag{F.2}$$

or equivalently

$$G(x,y) = -\frac{1}{(2\pi)^n}\int\frac{d^nk\ e^{\iota k(x-y)}}{k^2+\iota\varepsilon} \tag{F.3}$$

The Jacobian for the above infinitesimal transformation is field independent as long as θ depends on A_μ linearly and hence can be neglected as it is an overall factor in W.

Transformation (F.1) leads to a WT identity

$$
\begin{aligned}
<-\xi_o\int d^ny\ [\partial.A]\ \theta[A(y)] + \int J_\mu(x)\partial_x^\mu G(x,y)\ \theta[A(y)]d^nxd^ny \\
+ \int J^*(x)\iota e_o\ \phi(x)\ G(x,y)\ \theta\ [A(y)]\ d^nx\ d^ny
\end{aligned}
$$

$$+ \int J(x)(-i e_o)\phi^*(x)G(x,y)\theta\ A(y)]d^nxd^ny>$$

$$= 0 \tag{F.4}$$

We let $\theta\ [A(y)] = \frac{1}{2}\ \partial.A(y)\ \varepsilon(y)$ and compare coefficients of $\varepsilon(y)$ and thus get,

$$\frac{\xi_o}{2}[\partial.A(y)]^2 = \frac{1}{2}\ \int J_\mu(x)\partial_x^{\ \mu}G(x,y)\langle\partial.A(y)\rangle d^nx$$

$$+ \frac{i e_o}{2}\int J^*(x)G(x,y)\langle\phi(x)\partial.A(y)\rangle d^nx$$

$$- \frac{i e_o}{2}\int J(x)G(x,y)\langle\phi^*(x)\partial.A(y)\rangle d^nx \tag{F.5}$$

We note that $J_\mu = -\dfrac{\delta\Gamma}{\delta\mathcal{A}_\mu} = -\dfrac{\delta\Gamma}{\delta\mathcal{A}^R_\mu}\ Z_3^{-1/2}$

$$\langle O_\sigma\rangle = \langle\frac{\xi_o}{2}[\partial.A(y)]^2\rangle =$$

$$-\ \frac{1}{2}\int \frac{\delta\Gamma}{\delta\mathcal{A}_\mu(x)}Z_3^{-1/2}\partial_x^{\ \mu}G(x,y)\langle\partial.A(y)\rangle d^nx$$

$$-\ \frac{i e_o}{2}\int\frac{\delta\Gamma}{\delta\Phi^R(x)}Z^{-1/2}G(x,y)\langle\phi(x)\partial.A(y)\rangle d^nx$$

$$+\ \frac{i e_o}{2}\int\frac{\delta\Gamma}{\delta\Phi^{*R}(x)}Z^{-1/2}G(x,y)\langle\phi^*(x)\partial.A(y)\rangle d^nx \tag{F.6}$$

From (F.6) it follows that the divergence in the left hand side which must be local functional of dimension four must have the form[see F.N.2 of chapter 5],

$$\langle O_\sigma\rangle = \alpha(\varepsilon)\ \frac{\delta S}{\delta\mathcal{A}^R_\mu(y)}\ \mathcal{A}^R_\mu(y) + \beta(\varepsilon)\left[\frac{\delta S}{\delta\phi^R}\phi^R + \frac{\delta S}{\delta\phi^{*R}}\phi^{*R}\right] \tag{F.7}$$

and thus can mix with O_3 and O_4 only. Actually the term in (F.6) proportional to O_4 is finite since $Z_3^{-1/2}\langle A_\mu\rangle = \langle A_\mu^R\rangle$,

Next, we show that $\xi_o\langle\partial^\rho(\partial.A\ A_\rho)\rangle$ is a finite operator.

This has already been shown in the appendix of Ref.8 of chapter 5 in the context of non-abelian gauge theories (where there are additional ghost terms in the operator).But a simpler proof can be given in the present context from Eq.(F.4). We let

$$\theta[A(y)] = A_\rho \epsilon^\rho(y)$$

and compare coefficients of $\epsilon^\rho(y)$ to get

$$\frac{\xi_o}{z} [\partial.A(y)A_\rho(y)] = \int J_\mu(x)\ \partial_x^{\ \mu}G(x,y)\ \langle\ \partial.A_\rho(y)\ \rangle d^n x$$

$$+\ \iota e_o \int J^*(x)\ G(x,y)\ \langle\phi(x)\ \partial.A_\rho(y)\ \rangle d^n x$$

$$-\iota e_o \int J(x)G(x,y)\langle\phi^*(x)\partial.A_\rho(y)\rangle d^n x \qquad \text{(F.8)}$$

$$= -\int \frac{\delta\Gamma}{\delta \mathscr{A}^R_\mu(x)}\ \partial_x^{\ \mu}G(x,y)\ \langle\ \partial.A^R_\rho(y)\ \rangle d^n x$$

$$-\ \iota e_o \int \frac{\delta\Gamma}{\delta\Phi^R(x)}\ G(x,y)\langle Z^{-1/2}\phi(x)\partial.A_\rho(y)\rangle d^n x$$

$$+\ \iota e_o \int \frac{\delta\Gamma}{\delta\Phi^{*R}} G(x,y)\langle Z^{-1/2}\phi^*(x)\partial.A_\rho(y)\rangle d^n x \qquad \text{(F.9)}$$

As before the first term on the right hand side of Eq.(F.9) is finite and the divergence in the left hand side of Eq.(F.9) has the form

$$\left\{\xi_o(\partial.A(y)A_\rho(y)\rangle\right\}^{div} = \frac{\delta S}{\delta\phi^R}\ F_\rho[\phi,\phi^*,\mathscr{A}^R_\mu] + \frac{\delta S}{\delta\phi^{*R}}\ F^*_\rho[\phi,\phi^*,\mathscr{A}^R_\mu]$$

where F_ρ and F^*_ρ are local functionals. But $\partial.A\ A_\rho$ is a dimension three operator. Hence, F_ρ and F^*_ρ must be dimensionless. But the right hand side must be a globally U(1)-invariant operator. Hence, $F_\rho = F^*_\rho = 0$, as no such $F_\rho = F^*_\rho$ exists.

This proves the finiteness of $\xi_o(\partial . AA_\rho)$ and hence of $\xi_o \partial^\rho(\partial . AA_\rho)$.

APPENDIX G

In this appendix, we shall obtain the Eq.(G.11) used in Sec.[5.5].

Firstly, one may notice that all Z_{ij}'s except Z_{11} and Z_{15} are polynomials in both λ and e^2 . [Z_{11} and Z_{15} have terms proportional to e^4/λ because λ is not multiplicatively renormalizable. $\delta\lambda$ in Eq.(5.4) begins with $O(e^4)$ and therefore $\beta_1^{\lambda}/\lambda$ has terms proportional to e^4/λ also].Nevertheless, Z_{11} and Z_{15} can also be considered as polynomials in loop expansion parameter a. As apparent from Eq.(5.29), λ_0 and e_0^2 both appear with a and therefore the ratio e^4/λ is also proportional to a. Therefore, if one expands Z_{11} (or Z_{15}) in powers of a, only positive powers appear. Hence, Z_{ij}^{-1} exists, when Z_{ij} is considered as a power series in a .And therefore, as argued before, Z_{ij} satisfies the equation

$$Z_{ij}^{-1}\ \mu_1\frac{\partial}{\partial\mu_1} Z_{jk} = \gamma_{ik} = \text{finite} \tag{G.1}$$

Using the form of $\mathbb{Z}$ in Eq.(5.15), the above equation (for $i = 1$, $k = 7$) reduces to

$$Z_{11}^{-1}\ \mu_1\frac{\partial}{\partial\mu_1} Z_{17} + Z_{15}^{-1}\ \mu_1\frac{\partial}{\partial\mu_1} Z_{57} + Z_{17}^{-1}\ \mu_1\frac{\partial}{\partial\mu_1} Z_{77} = \gamma_{17} \tag{G.2}$$

It is straightforward to verify that

$$(Z^{-1})_{11} = \frac{Z_{55}}{Z_{11}Z_{55} - Z_{15}Z_{51}} \qquad (Z^{-1})_{15} = -\frac{Z_{15}}{Z_{11}Z_{55} - Z_{15}Z_{51}}$$

$$(Z^{-1})_{17} = -\frac{Z_m(Z_{55}Z_{17} - Z_{57}Z_{15})}{Z_{11}Z_{55} - Z_{15}Z_{51}} \tag{G.3}$$

Combining this with Eq.(G.2) one obtains

$$Z_{55}\mu_1 \frac{\partial}{\partial\mu_1} Z_{17} - Z_{15}\mu_1 \frac{\partial}{\partial\mu_1} Z_{57} - Z_m(Z_{55}Z_{17} - Z_{57}Z_{15})\ \mu_1 \frac{\partial}{\partial\mu_1} Z_m^{-1}$$
$$= \gamma_{17}(Z_{11}Z_{15} - Z_{15}Z_{51}) \tag{G.4}$$

Similarly, using

$$(Z^{-1}\ \mu_1 \frac{\partial}{\partial\mu_1} Z)_{57} = \gamma_{57} \tag{G.5}$$

and

$$(Z^{-1})_{51} = \frac{Z_{51}}{Z_{11}Z_{55} - Z_{15}Z_{51}} \qquad (Z^{-1})_{55} = \frac{Z_{11}}{Z_{11}Z_{55} - Z_{15}Z_{51}}$$

$$(Z^{-1})_{57} = -\frac{Z_m(Z_{11}Z_{57} - Z_{17}Z_{51})}{Z_{11}Z_{55} - Z_{15}Z_{51}} \tag{G.6}$$

one obtains

$$-Z_{51}\ \mu_1 \frac{\partial}{\partial\mu_1} Z_{17} + Z_{11}\ \mu_1 \frac{\partial}{\partial\mu_1} Z_{57} - Z_m(Z_{11}Z_{57} - Z_{17}Z_{15})\ \mu_1 \frac{\partial}{\partial\mu_1} Z_m^{-1}$$
$$= \gamma_{57}(Z_{11}Z_{55} - Z_{15}Z_{51}) \tag{G.7}$$

Multiplying Eq.(G.4) by Z_{11} and Eq.(G.7) by Z_{15} and adding, one obtains,

$$\mu_1 \frac{\partial}{\partial\mu_1} Z_{17} - 2\gamma_{m_1} Z_{17} = Z_{11}\gamma_{17} + Z_{15}\gamma_{57} \tag{G.8}$$

Where we have used Eq.(5.10).Similarly one can show that

$$\mu_2 \frac{\partial}{\partial\mu_2} Z_{17} - 2\gamma_{m_2} Z_{17} = Z_{11}\gamma'_{17} + Z_{15}\gamma'_{57} \tag{G.9}$$

where γ'_{17} and γ'_{57} are defined by

$$(Z^{-1}\mu_2 \frac{\partial}{\partial\mu_2} Z)_{ij} = \gamma'_{ij} \tag{G.10}$$

Now

$$\mu_1 \frac{\partial}{\partial\mu_1} Z_{17} = (-\lambda\varepsilon + \beta_1^\lambda)\frac{\partial Z_{17}}{\partial\lambda} + \beta_1^e \frac{\partial Z_{17}}{\partial e} + c\frac{\partial}{\partial c} Z_{17}$$

and

$$\mu_2 \frac{\partial}{\partial\mu_2} Z_{17} = (-\frac{e\varepsilon}{2} + \beta_2^e)\frac{\partial Z_{17}}{\partial e} + \beta_2^\lambda \frac{\partial Z_{17}}{\partial\lambda} - c\frac{\partial}{\partial c} Z_{17}$$

Hence Z_{17} satisfies the following equation,

$$\left(-\lambda\varepsilon+\beta_1^{\lambda}+\beta_2^{\lambda}\right)\frac{\partial Z_{17}}{\partial\lambda}+\left(-\frac{e\varepsilon}{2}+\beta_1^{e}+\beta_2^{e}\right)\frac{\partial Z_{17}}{\partial e}-2(\gamma_{m_1}+\gamma_{m_2})Z_{17}$$
$$=Z_{11}(\gamma_{17}+\gamma_{17}')+Z_{15}(\gamma_{57}+\gamma_{57}') \qquad (G.11)$$

APPENDIX H

In this appendix, we will show that the set of operators $O_\iota\,[\iota=1,2,\cdots,8]$ in Sec.[6.2] is closed under renormalization and that the operator

$$O' = \xi_o A^a_\mu(\partial . A^a) - \overline{C}^a D^{ab}_\mu C_b$$

is a finite operator.

Consider the operator

$$O = -\frac{1}{2}\,\xi_o \sum_a (\partial . A^a)^2 + \overline{C}^a\; D^{ab}_\mu C_b$$

which is invariant under BRS transformations. If one considers an action with a source term added that couples to O,

$$S' = S + \int d^n x\; N(x) O(x)$$

then S' is also BRS invariant. From this fact, it is easy to show that the WT identity satisfied by the divergent part of the generating functional for proper vertices with one insertion of O(x) is identical to that satisfied by the corresponding generating functional for a gauge invariant operator. Hence, it can only mix with those operators that mix with a dimension 4 Lorentz scalar operator. These operators are (O_1-O_7), O_2, O_3, O_4, (O_6-O_7) and O_8. Now, from equations of motion of antighost field it is easy to show that $\overline{C}^a\partial^\mu D^{ab}_\mu C_b$ is a finite operator. Further, $\langle -\frac{1}{2}\,\xi_o \sum_a (\partial . A^a)^2\rangle$ satisfies the Ward identity,

$$\begin{aligned}
&\langle -\frac{1}{2}\,\xi_o \sum_a (\partial . A^a)^2 + \overline{C}^a\partial^\mu D^{ab}_\mu C_b \rangle \\
&\quad = \iota \langle \overline{C}^a(x)\partial . A^a(x) \quad \int d^n x' [J^b_\mu(x') D^{bd}_\mu C_d(x') \\
&\quad + J^b(x') f_{bde}\phi_d(x') C_e(x') - \overline{\eta}_b(x')\frac{1}{2}\, e_o f_{bde}\; C_d(x') C_e(x')
\end{aligned}$$

$$-\frac{1}{2}\xi_o(\partial.A^b)\ \eta_b(x')]> \tag{H.1}$$

Note that each term on the right hand side is proportional to a source. Hence, the only operators from the above set consistent with Eq.(H.1) are O_3, O_4 and O_5. Hence, only Z_{73}, Z_{74} and Z_{75} are non-zero, while Z_{77} is 1.

The operator O_4 is a class I gauge-invariant operator[see Ref.8 of chapter 5]. It can mix only with class I operators (here O_3 and O_4) and with class II operators(here O_5). (O_1-O_7), O_2, (O_6-O_7) and O_8 are gauge-invariant operators which do not vanish by equations of motion and from the results about renormalization of gauge invariant operators they can only mix with themselves and operators O_3, O_4 and O_5. Hence, the set O_i is closed under renormalization.

Next, we shall show that the operator

$$O' = \xi_o A^a_\mu\ (\partial.A^a) - \overline{C}_a\ D^{ab}_\mu C_b$$

is a finite operator. This follows from the identity

$$\begin{aligned}&<-\frac{1}{2}\xi_o A^a_\mu(x)\partial.A^a(x) + \overline{C}^a(x)D^{ab}_\mu C_b(x)>\\ &\quad = i<\overline{C}_a(x)\ A^a_\mu(x)\int d^n x'\Sigma\ J^b_\mu(x')\ \ D^{bd}_\mu C_d(x')\\ &\quad +J^b(x')f_{bde}\phi_d(x')C_e(x') - \overline{\eta}_b(x')\frac{1}{2}\ e_o f_{bde}\ \ C_d(x')C_e(x')\\ &\quad -\frac{1}{2}\xi_o(\partial.A^b)\ \eta_b(x')]>\end{aligned} \tag{H.2}$$

This Ward identity is very similar to the one satisfied by $A_\mu(x)\partial.A(x)$ given in Appendix F for the scalar Q.E.D. case in that each term on the right hand side is proportional to a source. The argument following Eq.(F.8) based on dimensions and global gauge invariance applies here also and the right hand side of Eq.(H.2) is finite proving the finiteness of O'.

ERRATUM

On page 2-21, read L.H.S. of Eq.(2.57) as $\langle \partial_\mu J^\mu_5 \rangle$

On page 3-~~2~~7, read R.H.S of Eq.(3.5) as

$$\theta_{\mu\nu} + H_o(\partial_\mu \partial_\nu - g_{\mu\nu}\partial^2)\phi^2$$

On page 3-13, line # 11 replace a_{11} by a_{10}

On page 3-10 read Eq.(3.14) as

$$S = S_o - \frac{n-2}{8(n-1)} R\phi^2$$

On page 3-13 Eq.(3.22) replace L.H.S. by $\varkappa_o(\lambda, m^2/\mu^2, \epsilon)$

In footnote on page 4-9 replace $\theta^{\alpha\mu}$ by $\theta^{\alpha\mu}_{\mu}$

On page 4-12, last line , replace 4-42 by 4-43

On page 5-6 footnote the equations are

$$Z_e = Z^{-1/2} \quad \text{and} \quad Z_e = 1 + O(e^2)$$

On page 5-8 add

$$O_2 = m_o^2 \phi^* \phi \quad \text{to Eq.(5.11)}$$

On page 5-12 in Eq.(5.2) replace

$$\frac{1}{2} g^{\alpha\beta} (D_\alpha\phi)^*(D_\beta\phi) \quad \text{by} \quad g^{\alpha\beta} (D_\alpha\phi)^*(D_\beta\phi)$$

On page 5-12 read Eq.(5.22) as

$$\begin{aligned} \theta_{\mu\nu} = -g_{\mu\nu}\mathcal{L}_{eff}\phi - 2[\tfrac{1}{8}(F_{\mu\gamma} F_\nu{}^\gamma + F_\mu{}^\gamma F_{\nu\gamma}) + \tfrac{1}{8}(F_{\gamma\mu} F^\gamma{}_\nu + F^\gamma{}_\nu F_{\gamma\mu}] \\ + [(D_\mu\phi)^*(D_\nu\phi) + (D_\nu\phi)^*(D_\mu\phi)] - \tfrac{1}{2}\xi_o(\partial.A)^2 g_{\mu\nu} + \\ + \xi_o[A_\nu D_\mu(\partial.A) + A_\mu\partial_\nu(\partial.A)] - \xi_o g_{\mu\nu} A_\rho \partial_\rho(\partial.A) \end{aligned}$$

On page 5-13 read the last term on R.H.S. of Eq.(5.25) as $\partial^2(\phi^*\phi)$

On page 5-13 read the L.H.S. of Eq.(5.26) as $\langle \theta_\mu^{imp\ \mu} \rangle$

On page 5-14 replace " = finite " by " + finite " in Eq.(5.27)

On page 6.10 read L.H.S of Eq.(6.30) as θ_μ^{imp}

On page 6-15 read second line of F-3 as

(i) $\frac{\delta S}{\delta\psi}\psi$, $\bar\psi\frac{\delta S}{\delta\bar\psi}$ and $\frac{\delta S}{\delta\phi}\phi$ are finite operators.

On page 6-20 in Eq.(6.63) replace γ_{10} by γ_{16}.

On page A-1 L.H.S. of Eq.(A.2) reads $\phi_m^{\dagger}(x)\ [-\not\partial + \not A\,] = \lambda_m \phi_m^{\dagger}(x)$

On page B-2 in fifth line replace Eq.(2.40) by Eq.(2.45).

On page B-2 last equation replace $\left(-\frac{1}{4}\right)^{\gamma-3}$ by $\left(-\frac{1}{4}\right)^{\gamma-2}$ and $\left(-\frac{1}{4}\right)^{\gamma-2}$ by $\left(-\frac{1}{4}\right)^{\gamma-1}$.

On page B-14 read L.H.S. of B-14 as $O_{\mu\nu}^{(1)}(2n-2)$.

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